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On Implicit and Inverse Function Theorems on Euclidean Spaces[1]

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Summary. Previous Mizar articles [7, 6, 5] formalized the implicit and inverse function theorems for Frechet continuously differentiable maps on Banach spaces. In this paper, using the Mizar system [1], [2], we formalize these theorems on Euclidean spaces by specializing them. We referred to [4], [12], [10], [11] in this formalization.

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1. Matrix and Linear Transformation on Euclidean Spaces

Let n be a natural number. One can check that $\langle \mathcal{E}^n, \|\cdot\|\rangle$ is finite dimensional. Now we state the propositions:

(1) Let us consider a non zero natural number n, and a real normed space X. Then every linear operator from $\langle \mathcal{E}^n, \|\cdot\|\rangle$ into X is Lipschitzian.

(2) Let us consider a non zero natural number m, and finite sequences s, t of elements of $\mathcal{R}^m$. Suppose $1 \leqslant \operatorname{len} s$ and $s = t{\restriction}\operatorname{len} s$. Let us consider a natural number i. If $1 \leqslant i \leqslant \operatorname{len} s$, then $(\operatorname{accum} t)(i) = (\operatorname{accum} s)(i)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} s$, then $(\operatorname{accum} t)(\$_1) = (\operatorname{accum} s)(\$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

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(3) Let us consider a non zero natural number m, finite sequences s, s_1 of elements of $\mathcal{R}^m$, and an element s_0 of $\mathcal{R}^m$. If $s_1 = s \frown \langle s_0 \rangle$, then $\sum s_1 = \sum s + s_0$. The theorem is a consequence of (2).

(4) Let us consider a non zero natural number m, a finite sequence s of elements of $\mathcal{R}^m$, and a natural number j. Suppose $1 \leqslant j \leqslant m$. Then there exists a finite sequence t of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} t = \operatorname{len} s$, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} s$ there exists an element s_2 of $\mathcal{R}^m$ such that $s_2 = s(i)$ and $t(i) = s_2(j)$, and

(iii) $(\sum s)(j) = \sum t$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence s of elements of $\mathcal{R}^m$ for every natural number j such that $\operatorname{len} s = \$_1$ and $1 \leqslant j \leqslant m$ there exists a finite sequence t of elements of $\mathbb{R}$ such that $\operatorname{len} t = \operatorname{len} s$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} s$ there exists an element s_2 of $\mathcal{R}^m$ such that $s_2 = s(i)$ and $t(i) = s_2(j)$ and $(\sum s)(j) = \sum t$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(5) Let us consider a non zero natural number m, and an element x of $\mathcal{R}^m$. Then there exists a finite sequence s of elements of $\mathcal{R}^m$ such that

(i) $\operatorname{dom} s = \operatorname{Seg} m$, and

(ii) for every natural number i such that $1 \leqslant i \leqslant m$ there exists an element e of $\mathcal{R}^m$ such that $e = (\operatorname{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1)$ and $s(i) = (\operatorname{proj}(i, m))(x) \cdot e$, and

(iii) $\sum s = x$.

PROOF: Define $\mathcal{P}$[natural number, object] $\equiv$ there exists an element e of $\mathcal{R}^m$ such that $e = (\operatorname{reproj}(\$_1, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1)$ and $\$_2 = (\operatorname{proj}(\$_1, m))(x) \cdot e$. For every natural number i such that $i \in \operatorname{Seg} m$ there exists an element y of $\mathcal{R}^m$ such that $\mathcal{P}[i, y]$. Consider s being a finite sequence of elements of $\mathcal{R}^m$ such that $\operatorname{dom} s = \operatorname{Seg} m$ and for every natural number i such that $i \in \operatorname{Seg} m$ holds $\mathcal{P}[i, s(i)]$. For every natural number i such that $1 \leqslant i \leqslant m$ there exists an element e of $\mathcal{R}^m$ such that $e = (\operatorname{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1)$ and $s(i) = (\operatorname{proj}(i, m))(x) \cdot e$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} \sum s$ holds $(\sum s)(i) = x(i)$. $\square$

(6) Let us consider non zero elements m, n of $\mathbb{N}$, and a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times n$. Then $\operatorname{Mx2Tran}(M)$ is a Lipschitzian linear operator from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

PROOF: Reconsider $f = \text{Mx2Tran}(M)$ as a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. For every elements x, y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, $f(x+y) = f(x)+f(y)$. For every vector x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ and for every real number a, $f(a \cdot x) = a \cdot f(x)$ by [8, (4),(8)]. □

Let us consider a non zero element m of $\mathbb{N}$ and a linear operator f from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Now we state the propositions:

(7) Suppose f is bijective. Then there exists a Lipschitzian linear operator g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that

(i) $g = f^{-1}$, and

(ii) g is one-to-one and onto.

(8) Suppose f is bijective. Then there exists a point g of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that

(i) $g = f$, and

(ii) g is invertible.

The theorem is a consequence of (7).

Let us consider non zero elements m, n of $\mathbb{N}$ and a square matrix M over $\mathbb{R}_\text{F}$ of dimension m. Now we state the propositions:

(9) $\text{Mx2Tran}(M)$ is bijective if and only if $\text{Det}\, M \neq 0_{\mathbb{R}_\text{F}}$.

(10) $\text{Mx2Tran}(M)$ is bijective if and only if M is invertible.

(11) Let us consider a non zero element m of $\mathbb{N}$, and a point f of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose f is one-to-one and $\text{rng}\, f =$ the carrier of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Then f is invertible. The theorem is a consequence of (8).

Let us consider a non zero element m of $\mathbb{N}$, a point f of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and a square matrix M over $\mathbb{R}_\text{F}$ of dimension m. Now we state the propositions:

(12) If $f = \text{Mx2Tran}(M)$, then f is invertible iff M is invertible. The theorem is a consequence of (10) and (11).

(13) If $f = \text{Mx2Tran}(M)$, then f is invertible iff $\text{Det}\, M \neq 0_{\mathbb{R}_\text{F}}$. The theorem is a consequence of (12).

Let us consider non zero elements m, n of $\mathbb{N}$. Now we state the propositions:

(14) There exists a function f from $\mathcal{R}^m \times \mathcal{R}^n$ into $\mathcal{R}^{m+n}$ such that

(i) for every element x of $\mathcal{R}^m$ and for every element y of $\mathcal{R}^n$, $f(x, y) = x \frown y$, and

(ii) f is one-to-one and onto.

Proof: Define $\mathcal{S}[\text{object}, \text{object}, \text{object}] \equiv$ there exists an element x of $\mathcal{R}^m$ and there exists an element y of $\mathcal{R}^n$ such that $x = \$_1$ and $y = \$_2$ and $\$_3 = x \frown y$. For every objects x, y such that $x \in \mathcal{R}^m$ and $y \in \mathcal{R}^n$ there exists an object z such that $z \in \mathcal{R}^{m+n}$ and $\mathcal{S}[x, y, z]$. Consider f being a function from $\mathcal{R}^m \times \mathcal{R}^n$ into $\mathcal{R}^{m+n}$ such that for every objects x, y such that $x \in \mathcal{R}^m$ and $y \in \mathcal{R}^n$ holds $\mathcal{S}[x, y, f(x, y)]$. For every element x of $\mathcal{R}^m$ and for every element y of $\mathcal{R}^n$, $f(x, y) = x \frown y$. □

(15) There exists a function f from $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{m+n}, \|\cdot\| \rangle$ such that

(i) f is one-to-one and onto, and

(ii) for every element x of $\mathcal{R}^m$ and for every element y of $\mathcal{R}^n$, $f(x, y) = x \frown y$, and

(iii) for every points u, v of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$, $f(u+v) = f(u) + f(v)$, and

(iv) for every point u of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every real number r, $f(r \cdot u) = r \cdot f(u)$, and

(v) $f(0_{\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle}) = 0_{\langle \mathcal{E}^{m+n}, \|\cdot\| \rangle}$, and

(vi) for every point u of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$, $\|f(u)\| = \|u\|$.

Proof: Consider f being a function from $\mathcal{R}^m \times \mathcal{R}^n$ into $\mathcal{R}^{m+n}$ such that for every element x of $\mathcal{R}^m$ and for every element y of $\mathcal{R}^n$, $f(x, y) = x \frown y$ and f is one-to-one and onto. For every points u, v of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$, $f(u+v) = f(u) + f(v)$. For every point u of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every real number r, $f(r \cdot u) = r \cdot f(u)$. For every point u of $\langle \mathcal{E}^m, \|\cdot\| \rangle \times \langle \mathcal{E}^n, \|\cdot\| \rangle$, $\|f(u)\| = \|u\|$ by [9, (18)]. □

2. Total Derivative and Partial Derivative

Now we state the propositions:

(16) Let us consider real normed spaces X, Y, a point x of X, and a Lipschitzian linear operator f from X into Y. Then

(i) f is differentiable in x, and

(ii) $f = f'(x)$.

Proof: Set $C = \Omega_X$. Reconsider $g = (\text{the carrier of } X) \longmapsto 0_Y$ as a partial function from X to Y. Reconsider $f_0 = f$ as an element of $\text{BdLinOps}(X, Y)$. For every (0_X)-convergent sequence h of X such that h is non-zero holds $\|h\|^{-1} \cdot (g_* h)$ is convergent and $\lim(\|h\|^{-1} \cdot (g_* h)) = 0_Y$. For every point x_0 of X such that $x_0 \in C$ holds $f_{/x_0} - f_{/x} = f_0(x_0 - x) + g_{/x_0 - x}$. □

(17) Let us consider a non zero natural number n, a natural number i, and a point x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $1 \leqslant i \leqslant n$. Then

(i) $\mathrm{Proj}(i, n)$ is differentiable in x, and

(ii) $(\mathrm{Proj}(i, n))'(x) = \mathrm{Proj}(i, n)$.

The theorem is a consequence of (16).

Let us consider non zero natural numbers m, n, a partial function f from $\mathcal{R}^m$ to $\mathcal{R}^n$, and an element x of $\mathcal{R}^m$. Now we state the propositions:

(18) f is differentiable in x if and only if for every natural number i such that $1 \leqslant i \leqslant n$ there exists a partial function f_1 from $\mathcal{R}^m$ to $\mathcal{R}^1$ such that $f_1 = (\mathrm{Proj}(i, n)) \cdot f$ and f_1 is differentiable in x.

(19) f is differentiable in x if and only if for every natural number i such that $1 \leqslant i \leqslant n$ there exists a partial function f_1 from $\mathcal{R}^m$ to $\mathbb{R}$ such that $f_1 = (\mathrm{proj}(i, n)) \cdot f$ and f_1 is differentiable in x.

PROOF: For every natural number i, $\langle (\mathrm{proj}(i, n)) \cdot f \rangle = (\mathrm{Proj}(i, n)) \cdot f$ by [3, (11)]. For every natural number i such that $1 \leqslant i \leqslant n$ there exists a partial function F_1 from $\mathcal{R}^m$ to $\mathcal{R}^1$ such that $F_1 = (\mathrm{Proj}(i, n)) \cdot f$ and F_1 is differentiable in x. $\square$

(20) Let us consider non zero natural numbers m, n, a partial function f from $\mathcal{R}^m$ to $\mathcal{R}^n$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider a natural number i, and a partial function f_1 from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose $1 \leqslant i \leqslant n$ and $f_1 = (\mathrm{proj}(i, n)) \cdot f$. Then

(i) f_1 is differentiable in x, and

(ii) ${f_1}'(x) = (\mathrm{proj}(i, n)) \cdot (f'(x))$.

The theorem is a consequence of (19).

(21) Let us consider non zero natural numbers m, n, a partial function f from $\mathcal{R}^m$ to $\mathcal{R}^n$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider natural numbers i, j. Suppose $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then f is partially differentiable in x w.r.t. i and j. The theorem is a consequence of (19).

(22) Let us consider non zero natural numbers m, n, a partial function f from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and an element x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose f is differentiable in x. Let us consider natural numbers i, j. Suppose $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Then f is partially differentiable in x w.r.t. i and j.

(23) Let us consider a non zero natural number m, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider elements u, v of $\mathcal{R}^m$. Then $(f'(x))(u+v) = (f'(x))(u)+(f'(x))(v)$.

(24) Let us consider a non zero natural number m, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider an element u of $\mathcal{R}^m$, and a real number a. Then $(f'(x))(a \cdot u) = a \cdot (f'(x))(u)$.

(25) Let us consider a non zero natural number m, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider a finite sequence s of elements of $\mathcal{R}^m$, and a finite sequence t of elements of $\mathbb{R}$. Suppose $\operatorname{dom} s = \operatorname{dom} t$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $t(i) = (f'(x))(s(i))$. Then $(f'(x))(\sum s) = \sum t$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence s of elements of $\mathcal{R}^m$ for every finite sequence t of elements of $\mathbb{R}$ such that $\operatorname{len} s = \$_1$ and $\operatorname{dom} s = \operatorname{dom} t$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $t(i) = (f'(x))(s(i))$ holds $(f'(x))(\sum s) = \sum t$. $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(26) Let us consider a non zero natural number m, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Let us consider an element d_1 of $\mathcal{R}^m$. Then there exists a finite sequence d_2 of elements of $\mathbb{R}$ such that

(i) $\operatorname{dom} d_2 = \operatorname{Seg} m$, and

(ii) for every natural number i such that $1 \leqslant i \leqslant m$ holds $d_2(i) = (\operatorname{proj}(i,m))(d_1) \cdot (\operatorname{partdiff}(f,x,i))$, and

(iii) $(f'(x))(d_1) = \sum d_2$.

Proof: Consider s being a finite sequence of elements of $\mathcal{R}^m$ such that $\operatorname{dom} s = \operatorname{Seg} m$ and for every natural number i such that $1 \leqslant i \leqslant m$ there exists an element e of $\mathcal{R}^m$ such that $e = (\operatorname{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1)$ and $s(i) = (\operatorname{proj}(i,m))(d_1) \cdot e$ and $\sum s = d_1$. Define $\mathcal{F}(\text{natural number}) = (f'(x))(s(\$_1))(\in \mathbb{R})$. Consider d_2 being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} d_2 = m$ and for every natural number i such that $i \in \operatorname{dom} d_2$ holds $d_2(i) = \mathcal{F}(i)$. For every natural number i such that $i \in \operatorname{dom} d_2$ holds $d_2(i) = (f'(x))(s(i))$. For every natural number i such that $1 \leqslant i \leqslant m$ holds $d_2(i) = (\operatorname{proj}(i,m))(d_1) \cdot (\operatorname{partdiff}(f,x,i))$. □

(27) Let us consider non zero elements m, n of $\mathbb{N}$, a subset X of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and a partial function f from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose X is open and $X \subseteq \operatorname{dom} f$. Then f is differentiable on X and $f'_{\restriction X}$ is continuous on X if and only if for every natural numbers i, j such that $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ holds $(\operatorname{Proj}(j,n)) \cdot f$ is partially differentiable on X w.r.t. i and $(\operatorname{Proj}(j,n)) \cdot f {\restriction^i} X$ is continuous on X.

Proof: For every natural number i such that $1 \leqslant i \leqslant m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^{i}X$ is continuous on X. □

3. Jacobian Matrix

Let m, n be non zero natural numbers, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. The functor Jacobian(f, x) yielding a matrix over $\mathbb{R}_{\rm F}$ of dimension $m \times n$ is defined by

(Def. 1) for every natural numbers i, j such that $i \in \operatorname{Seg} m$ and $j \in \operatorname{Seg} n$ holds $it_{i,j} = \operatorname{partdiff}(f, x, i, j)$.

Now we state the proposition:

(28) Let us consider non zero natural numbers m, n, a partial function f from $\mathcal{R}^m$ to $\mathcal{R}^n$, and an element x of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x) = \operatorname{Mx2Tran}(\operatorname{Jacobian}(f, x))$.
Proof: For every element d_1 of $\mathcal{R}^m$, $(f'(x))(d_1) = (\operatorname{Mx2Tran}(\operatorname{Jacobian}(f, x)))(d_1)$. □

Let m, n be non zero natural numbers, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and x be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. The functor Jacobian$(f, x)$ yielding a matrix over $\mathbb{R}_{\rm F}$ of dimension $m \times n$ is defined by

(Def. 2) there exists a partial function g from $\mathcal{R}^m$ to $\mathcal{R}^n$ and there exists an element y of $\mathcal{R}^m$ such that $g = f$ and $y = x$ and $it = \operatorname{Jacobian}(g, y)$.

Now we state the proposition:

(29) Let us consider non zero elements m, n of $\mathbb{N}$, a point x of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and a partial function f from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose f is differentiable in x. Then $f'(x) = \operatorname{Mx2Tran}(\operatorname{Jacobian}(f, x))$. The theorem is a consequence of (28).

Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and a point x of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Now we state the propositions:

(30) If f is differentiable in x, then $f'(x)$ is invertible iff Jacobian(f, x) is invertible. The theorem is a consequence of (29) and (12).

(31) If f is differentiable in x, then $f'(x)$ is invertible iff $\operatorname{Det} \operatorname{Jacobian}(f, x) \neq 0_{\mathbb{R}_{\rm F}}$. The theorem is a consequence of (30).

4. Implicit and Inverse Function Theorems on Euclidean Spaces

Now we state the propositions:

(32) Let us consider non zero elements l, m, n of $\mathbb{N}$, a subset Z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$, a partial function f from $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a point a of $\langle \mathcal{E}^l, \|\cdot\| \rangle$, a point b of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, a point c of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a point z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $\langle\!\langle a, b \rangle\!\rangle \in Z$ and $f(a, b) = c$ and $z = \langle\!\langle a, b \rangle\!\rangle$ and $\mathrm{partdiff}(f, z)$ w.r.t. 2 is invertible. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\mathrm{Ball}(a, r_1) \times \overline{\mathrm{Ball}}(b, r_2) \subseteq Z$, and

(iv) for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $y \in \mathrm{Ball}(b, r_2)$ and $f(x, y) = c$, and

(v) for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ for every points y_1, y_2 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $y_1, y_2 \in \mathrm{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$, and

(vi) there exists a partial function g from $\langle \mathcal{E}^l, \|\cdot\| \rangle$ to $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $\mathrm{dom}\, g = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(b, r_2)$ and g is continuous on $\mathrm{Ball}(a, r_1)$ and $g(a) = b$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and g is differentiable on $\mathrm{Ball}(a, r_1)$ and $g'_{\restriction \mathrm{Ball}(a, r_1)}$ is continuous on $\mathrm{Ball}(a, r_1)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ and for every point z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ and $z = \langle\!\langle x, g(x) \rangle\!\rangle$ holds $g'(x) = -(\mathrm{Inv}\, \mathrm{partdiff}(f, z)$ w.r.t. 2$) \cdot (\mathrm{partdiff}(f, z)$ w.r.t. 1$)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ and for every point z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ and $z = \langle\!\langle x, g(x) \rangle\!\rangle$ holds $\mathrm{partdiff}(f, z)$ w.r.t. 2 is invertible, and

(vii) for every partial functions g_1, g_2 from $\langle \mathcal{E}^l, \|\cdot\| \rangle$ to $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $\mathrm{dom}\, g_1 = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Ball}(b, r_2)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(b, r_2)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\| \rangle$ such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

(33) Let us consider non zero elements l, m of $\mathbb{N}$, a subset Z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$, a partial function f from $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^m, \|\cdot\| \rangle$, a point a of $\langle \mathcal{E}^l, \|\cdot\| \rangle$, points b, c of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and a point z of $\langle \mathcal{E}^l, \|\cdot\| \rangle \times \langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable on

Z and $f'_{\restriction Z}$ is continuous on Z and $\langle a, b\rangle \in Z$ and $f(a,b) = c$ and $z = \langle a, b\rangle$ and Det Jacobian$(f \cdot (\text{reproj2}(z)), (z)_{\mathbf{2}}) \neq 0_{\mathbb{R}_{\mathrm{F}}}$. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\text{Ball}(a, r_1) \times \overline{\text{Ball}}(b, r_2) \subseteq Z$, and

(iv) for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ there exists a point y of $\langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $y \in \text{Ball}(b, r_2)$ and $f(x, y) = c$, and

(v) for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ for every points y_1, y_2 of $\langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $y_1, y_2 \in \text{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$, and

(vi) there exists a partial function g from $\langle \mathcal{E}^l, \|\cdot\|\rangle$ to $\langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $\text{dom}\, g = \text{Ball}(a, r_1)$ and $\text{rng}\, g \subseteq \text{Ball}(b, r_2)$ and g is continuous on $\text{Ball}(a, r_1)$ and $g(a) = b$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and g is differentiable on $\text{Ball}(a, r_1)$ and $g'_{\restriction \text{Ball}(a, r_1)}$ is continuous on $\text{Ball}(a, r_1)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ and for every point z of $\langle \mathcal{E}^l, \|\cdot\|\rangle \times \langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ and $z = \langle x, g(x)\rangle$ holds $g'(x) = -(\text{Inv partdiff}(f, z)\text{ w.r.t. } 2) \cdot (\text{partdiff}(f, z)\text{ w.r.t. } 1)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ and for every point z of $\langle \mathcal{E}^l, \|\cdot\|\rangle \times \langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ and $z = \langle x, g(x)\rangle$ holds $\text{partdiff}(f, z)\text{ w.r.t. } 2$ is invertible, and

(vii) for every partial functions g_1, g_2 from $\langle \mathcal{E}^l, \|\cdot\|\rangle$ to $\langle \mathcal{E}^m, \|\cdot\|\rangle$ such that $\text{dom}\, g_1 = \text{Ball}(a, r_1)$ and $\text{rng}\, g_1 \subseteq \text{Ball}(b, r_2)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\text{dom}\, g_2 = \text{Ball}(a, r_1)$ and $\text{rng}\, g_2 \subseteq \text{Ball}(b, r_2)$ and for every point x of $\langle \mathcal{E}^l, \|\cdot\|\rangle$ such that $x \in \text{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

The theorem is a consequence of (31).

(34) Let us consider a non zero element m of $\mathbb{N}$, a subset Z of $\langle \mathcal{E}^m, \|\cdot\|\rangle$, a partial function f from $\langle \mathcal{E}^m, \|\cdot\|\rangle$ to $\langle \mathcal{E}^m, \|\cdot\|\rangle$, a point a of $\langle \mathcal{E}^m, \|\cdot\|\rangle$, and a point b of $\langle \mathcal{E}^m, \|\cdot\|\rangle$. Suppose Z is open and $\text{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $a \in Z$ and $f(a) = b$ and Det Jacobian$(f, a) \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

Then there exists a subset A of $\langle \mathcal{E}^m, \|\cdot\|\rangle$ and there exists a subset B of $\langle \mathcal{E}^m, \|\cdot\|\rangle$ and there exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\|\rangle$ to $\langle \mathcal{E}^m, \|\cdot\|\rangle$ such that A is open and B is open and $A \subseteq \text{dom}\, f$ and $a \in A$ and $b \in B$ and $f^{\circ} A = B$ and $\text{dom}\, g = B$ and $\text{rng}\, g = A$ and $\text{dom}(f \restriction A) = A$ and $\text{rng}(f \restriction A) = B$ and $f \restriction A$ is one-to-one and g is one-to-one and $g = (f \restriction A)^{-1}$

and $f\restriction A = g^{-1}$ and $g(b) = a$ and g is continuous on B and differentiable on B and $g'_{\restriction B}$ is continuous on B and for every point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $y \in B$ holds $f'(g_{/y})$ is invertible and for every point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $y \in B$ holds $g'(y) = \operatorname{Inv} f'(g_{/y})$. The theorem is a consequence of (31).

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