FORMALIZED MATHEMATICS
Vol. 30, No. 2, Pages 125–134, 2022
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0010
sciendo.com/journal/forma

Definition of Centroid Method as Defuzzification

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Summary. In this study, using the Mizar system [1], [2], we reuse formalization efforts in fuzzy sets described in [5] and [6]. This time the centroid method which is one of the fuzzy inference processes is formulated [10]. It is the most popular of all defuzzied methods ([11], [13], [7]) – here, defuzzified crisp value is obtained from domain of membership function as weighted average [8]. Since the integral is used in centroid method, the integrability and bounded properties of membership functions are also mentioned to fill the formalization gaps present in the Mizar Mathematical Library, as in the case of another fuzzy operators [4]. In this paper, the properties of piecewise linear functions consisting of two straight lines are mainly described.

MSC: 68V20 93C42

Keywords: defuzzification; centroid; piecewise linear function

MML identifier: FUZZY_6, version: 8.1.12 5.71.1431

From now on A denotes a non empty, closed interval subset of $\mathbb{R}$.

Let A be a non empty, closed interval subset of $\mathbb{R}$ and f be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor centroid(f, A) yielding a real number is defined by the term

(Def. 1) $$\frac{\int\limits_{A} (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx}{\int\limits_{A} f(x)dx}.$$

Now we state the propositions:

(1) Let us consider real numbers a, b, c. Suppose $a < b$ and $c > 0$. Then centroid(AffineMap$(0, c), [a, b]) = \frac{a+b}{2}$.

PROOF: Set $F = \frac{c}{2} \cdot (\square^2)$. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}(F'_{\restriction \Omega_{\mathbb{R}}})$ holds $(F'_{\restriction \Omega_{\mathbb{R}}})(x) = (\mathrm{id}_{\mathbb{R}} \cdot (\mathrm{AffineMap}(0, c)))(x)$ by [12, (2)]. For every element x of $\mathbb{R}$ such that $x \in \mathrm{dom}((\mathrm{AffineMap}(c, 0))'_{\restriction \Omega_{\mathbb{R}}})$ holds $((\mathrm{AffineMap}(c, 0))'_{\restriction \Omega_{\mathbb{R}}})(x) = (\mathrm{AffineMap}(0, c))(x)$. $\square$

(2) Let us consider real numbers a, b. Then

(i) $\mathrm{id}_{\mathbb{R}}$ is integrable on $[a, b]$, and

(ii) $\mathrm{id}_{\mathbb{R}} \restriction [a, b]$ is bounded.

(3) (i) $\mathrm{id}_{\mathbb{R}}$ is integrable on A, and

(ii) $\mathrm{id}_{\mathbb{R}} \restriction A$ is bounded.

(4) Let us consider a real number e, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A \subseteq \mathrm{dom}\, f$ and for every real number x such that $x \in A$ holds $f(x) = e$. Then

(i) f is integrable on A, and

(ii) $f \restriction A$ is bounded, and

(iii) $\int\limits_{\inf A}^{\sup A} f(x) dx = e \cdot (\sup A - \inf A)$.

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(5) If for every real number x such that $x \in A$ holds $f(x) = 0$, then $\int\limits_{A} f(x) dx = 0$. The theorem is a consequence of (4).

(6) Suppose f is integrable on A and $f \restriction A$ is bounded. Then

(i) $\mathrm{id}_{\mathbb{R}} \cdot f$ is integrable on A, and

(ii) $(\mathrm{id}_{\mathbb{R}} \cdot f) \restriction A$ is bounded.

The theorem is a consequence of (3).

(7) Let us consider real numbers a, b, c. Suppose $a < b$. Then

(i) $[a, b] \subseteq \Omega_{\mathbb{R}}$, and

(ii) $\inf[a, b] = a$, and

(iii) $\sup[a, b] = b$.

Let us consider real numbers a, b, c and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(8) Suppose $a < b \leqslant c$ and f is integrable on $[a, c]$ and $f \restriction [a, c]$ is bounded and for every real number x such that $x \in [b, c]$ holds $f(x) = 0$. Then $\mathrm{centroid}(f, [a, c]) = \mathrm{centroid}(f, [a, b])$. The theorem is a consequence of (3).

(9) Suppose $a \leqslant b < c$ and f is integrable on $[a, c]$ and $f \restriction [a, c]$ is bounded and for every real number x such that $x \in [a, b]$ holds $f(x) = 0$. Then $\mathrm{centroid}(f, [a, c]) = \mathrm{centroid}(f, [b, c])$. The theorem is a consequence of (3).

(10) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f \restriction A$ is bounded and $\int\limits_A f(x)dx > 0$. Then there exists a real number c such that

(i) $c \in A$, and

(ii) $f(c) > 0$.

PROOF: Set $g = (-1) \cdot f$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(g \restriction A)$ holds $|(g \restriction A)(y)| < r$. For every real number x such that $x \in A$ holds $0 \leqslant (g \restriction A)(x)$. □

(11) Let us consider a real number r, a fuzzy set f of $\mathbb{R}$, and a function F from $\mathbb{R}$ into $\mathbb{R}$. Suppose $r > 0$ and f is integrable on A and $f \restriction A$ is bounded and for every real number x, $F(x) = \min(r, f(x))$. Then $\int\limits_A F(x)dx \geqslant 0$.

PROOF: There exists a real number r such that for every set y such that $y \in \mathrm{dom}(F \restriction A)$ holds $|(F \restriction A)(y)| < r$. For every real number x such that $x \in A$ holds $0 \leqslant (F \restriction A)(x)$. □

Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(12) $\min(f, g) = \frac{1}{2} \cdot (f + g - |f - g|)$.
PROOF: For every object x such that $x \in \mathrm{dom}(\min(f, g))$ holds $(\min(f, g))(x) = (\frac{1}{2} \cdot (f + g - |f - g|))(x)$. □

(13) Suppose f is integrable on A and $f \restriction A$ is bounded and g is integrable on A and $g \restriction A$ is bounded. Then

(i) $\min(f, g)$ is integrable on A, and

(ii) $\min(f, g) \restriction A$ is bounded, and

(iii) $\int\limits_A (\min(f, g))(x)dx = \frac{1}{2} \cdot (\int\limits_A f(x)dx + \int\limits_A g(x)dx - \int\limits_A |f - g|(x)dx)$.

The theorem is a consequence of (12).

(14) $\max(f, g) = \frac{1}{2} \cdot (f + g + |f - g|)$.
PROOF: For every object x such that $x \in \mathrm{dom}(\max(f, g))$ holds $(\max(f, g))(x) = (\frac{1}{2} \cdot (f + g + |f - g|))(x)$. □

(15) Suppose f is integrable on A and $f \restriction A$ is bounded and g is integrable on A and $g \restriction A$ is bounded. Then

(i) $\max(f, g)$ is integrable on A, and

(ii) $\max(f,g)\restriction A$ is bounded, and

(iii) $\int\limits_A (\max(f,g))(x)dx = \frac{1}{2}\cdot(\int\limits_A f(x)dx + \int\limits_A g(x)dx + \int\limits_A |f-g|(x)dx)$.

The theorem is a consequence of (14).

(16) Let us consider real numbers r_1, r_2, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f\restriction A$ is bounded. Then

(i) $\min(\mathrm{AffineMap}(0,r_1), r_2 \cdot f)$ is integrable on A, and

(ii) $\min(\mathrm{AffineMap}(0,r_1), r_2 \cdot f)\restriction A$ is bounded.

The theorem is a consequence of (13).

(17) Let us consider real numbers r_1, r_2, and functions f, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is integrable on A and $f\restriction A$ is bounded and for every real number x, $F(x) = \min(r_1, r_2 \cdot f(x))$. Then

(i) F is integrable on A, and

(ii) $F\restriction A$ is bounded.

The theorem is a consequence of (16).

(18) Let us consider a real number s, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Then $f\restriction]-\infty,s[+\cdot g\restriction[s,+\infty[$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let us consider real numbers a, b, c and functions f, g, F from $\mathbb{R}$ into $\mathbb{R}$.

(19) If $a \leqslant b \leqslant c$ and $F = f\restriction[a,b]+\cdot g\restriction[b,c]$, then F is a function from $[a,c]$ into $\mathbb{R}$.

(20) If $a \leqslant b \leqslant c$ and $F = f\restriction[a,b]+\cdot g\restriction[b,c]$, then $F = F\restriction[a,c]$.

Let us consider real numbers a, b, c and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$.

(21) Suppose $a \leqslant b \leqslant c$ and $f\restriction[a,c]$ is bounded and $g\restriction[a,c]$ is bounded and $h = f\restriction[a,b]+\cdot g\restriction[b,c]$ and $f(b) = g(b)$. Then $h\restriction[a,c]$ is bounded.
PROOF: $f\restriction[a,b]$ tolerates $g\restriction[b,c]$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(h\restriction[a,c])$ holds $|(h\restriction[a,c])(y)| < r$. $\square$

(22) Suppose $a \leqslant b \leqslant c$ and $f\restriction[a,c]$ is bounded and $g\restriction[a,c]$ is bounded and $h\restriction[a,c] = f\restriction[a,b]+\cdot g\restriction[b,c]$ and $f(b) = g(b)$. Then $h\restriction[a,c]$ is bounded.
PROOF: $f\restriction[a,b]$ tolerates $g\restriction[b,c]$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(h\restriction[a,c])$ holds $|(h\restriction[a,c])(y)| < r$. $\square$

Now we state the propositions:

(23) Let us consider a real number c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f\restriction A$ is bounded and $g\restriction A$ is bounded. Then $(f\restriction]-\infty,c[+\cdot g\restriction[c,+\infty[)\restriction A$ is bounded.
PROOF: Set $F = f\restriction]-\infty,c[+\cdot g\restriction[c,+\infty[$. There exists a real number r such that for every set y such that $y \in \mathrm{dom}(F\restriction A)$ holds $|(F\restriction A)(y)| < r$. $\square$

(24) Let us consider real numbers a, b, c, and functions f, g, h, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h{\restriction}[a,c] = f{\restriction}[a,b]{+\cdot}g{\restriction}[b,c]$ and $f(b) = g(b)$ and $F = h{\restriction}[a,c]$. Then F is continuous.

PROOF: For every real numbers x_0, r such that $x_0 \in [a,c]$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number x_1 such that $x_1 \in [a,c]$ and $|x_1 - x_0| < s$ holds $|h(x_1) - h(x_0)| < r$. $\square$

(25) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is continuous. Then

(i) f is integrable on A, and

(ii) $f{\restriction}A$ is bounded.

(26) Let us consider a real number c, and functions f, g, F from $\mathbb{R}$ into $\mathbb{R}$. Suppose f is Lipschitzian and g is Lipschitzian and $f(c) = g(c)$ and $F = f{\restriction}]{-\infty}, c[{+\cdot}g{\restriction}[c, +\infty[$. Then F is Lipschitzian.

PROOF: Consider r_3 being a real number such that $0 < r_3$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r_3 \cdot |x_1 - x_2|$. Consider r_4 being a real number such that $0 < r_4$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} g$ holds $|g(x_1)-g(x_2)| \leqslant r_4 \cdot |x_1-x_2|$. There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} F$ holds $|F(x_1) - F(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

(27) Let us consider real numbers a, b. Then $\mathrm{AffineMap}(a,b)$ is Lipschitzian.

PROOF: Set $f = \mathrm{AffineMap}(a,b)$. There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

Let us consider real numbers a, b, p, q and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(28) Suppose $a \neq p$ and $f = (\mathrm{AffineMap}(a,b)){\restriction}]{-\infty}, \frac{q-b}{a-p}[{+\cdot}(\mathrm{AffineMap}(p,q)){\restriction}[\frac{q-b}{a-p}, +\infty[$. Then f is Lipschitzian. The theorem is a consequence of (27) and (26).

(29) Suppose $a \neq p$ and $f = (\mathrm{AffineMap}(a,b)){\restriction}]{-\infty}, \frac{q-b}{a-p}[{+\cdot}(\mathrm{AffineMap}(p,q)){\restriction}[\frac{q-b}{a-p}, +\infty[$. Then

(i) f is integrable on A, and

(ii) $f{\restriction}A$ is bounded.

The theorem is a consequence of (28).

(30) Let us consider real numbers a, b, p, q. Suppose $a \neq p$. Then $(\mathrm{AffineMap}(a,b))(\frac{q-b}{a-p}) = (\mathrm{AffineMap}(p,q))(\frac{q-b}{a-p})$.

(31) Every membership function of $\mathbb{R}$ is bounded.

PROOF: There exists a real number r such that for every set x such that $x \in \operatorname{dom} f$ holds $|f(x)| < r$ by [9, (1)]. □

(32) Let us consider a real number r, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $r \neq 0$ and f is integrable on A and $f\restriction A$ is bounded. Then $\operatorname{centroid}(r \cdot f, A) = \operatorname{centroid}(f, A)$. The theorem is a consequence of (6).

Let us consider real numbers a, b, c and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$.

(33) Suppose $a \leqslant b \leqslant c$ and f is integrable on $[a, c]$ and $f\restriction[a, c]$ is bounded and g is integrable on $[a, c]$ and $g\restriction[a, c]$ is bounded and $h\restriction[a, c] = f\restriction[a, b]{+\cdot}g\restriction[b, c]$ and h is integrable on $[a, c]$ and $f(b) = g(b)$.

Then $\int\limits_{[a,c]} h(x)dx = \int\limits_{[a,b]} f(x)dx + \int\limits_{[b,c]} g(x)dx$.

PROOF: $f\restriction[a, b]$ tolerates $g\restriction[b, c]$. Reconsider $h_1 = h\restriction[a, b]$ as a partial function from $[a, b]$ to $\mathbb{R}$. Reconsider $f_1 = f\restriction[a, b]$ as a partial function from $[a, b]$ to $\mathbb{R}$. Reconsider $H = \text{upper\_sum\_set}\, h_1$ as a function from $\operatorname{divs}[a, b]$ into $\mathbb{R}$. Reconsider $F = \text{upper\_sum\_set}\, f_1$ as a function from $\operatorname{divs}[a, b]$ into $\mathbb{R}$. $H = F$.

Reconsider $h_2 = h\restriction[b, c]$ as a partial function from $[b, c]$ to $\mathbb{R}$. Reconsider $g_1 = g\restriction[b, c]$ as a partial function from $[b, c]$ to $\mathbb{R}$. Reconsider $H_1 = \text{upper\_sum\_set}\, h_2$ as a function from $\operatorname{divs}[b, c]$ into $\mathbb{R}$. Reconsider $G = \text{upper\_sum\_set}\, g_1$ as a function from $\operatorname{divs}[b, c]$ into $\mathbb{R}$. $H_1 = G$. $h\restriction[a, c]$ is bounded. □

(34) Suppose $a \leqslant b \leqslant c$ and f is continuous and g is continuous and $h = f\restriction[a, b]{+\cdot}g\restriction[b, c]$ and $f(b) = g(b)$.

Then $\int\limits_{[a,c]} (\mathrm{id}_{\mathbb{R}} \cdot h)(x)dx = \int\limits_{[a,b]} (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx + \int\limits_{[b,c]} (\mathrm{id}_{\mathbb{R}} \cdot g)(x)dx$.

PROOF: $\mathrm{id}_{\mathbb{R}} \cdot f$ is integrable on $[a, c]$ and $(\mathrm{id}_{\mathbb{R}} \cdot f)\restriction[a, c]$ is bounded and $\mathrm{id}_{\mathbb{R}} \cdot g$ is integrable on $[a, c]$ and $(\mathrm{id}_{\mathbb{R}} \cdot g)\restriction[a, c]$ is bounded. Set $G = (\mathrm{id}_{\mathbb{R}} \cdot f)\restriction[a, b]{+\cdot}(\mathrm{id}_{\mathbb{R}} \cdot g)\restriction[b, c]$. For every object x such that $x \in \operatorname{dom} G$ holds $G(x) = (\mathrm{id}_{\mathbb{R}} \cdot h)(x)$. $\mathrm{id}_{\mathbb{R}} \cdot h$ is integrable on $[a, c]$. □

Let us consider a real number c and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(35) $f\restriction]-\infty, c[{+\cdot}g\restriction[c, +\infty[= f\restriction]-\infty, c]{+\cdot}g\restriction[c, +\infty[$.

PROOF: Set $f_1 = f\restriction]-\infty, c[{+\cdot}g\restriction[c, +\infty[$. Set $f_2 = f\restriction]-\infty, c]{+\cdot}g\restriction[c, +\infty[$. For every object x such that $x \in \operatorname{dom} f_1$ holds $f_1(x) = f_2(x)$. □

(36) Suppose $f\restriction A$ is bounded and $g\restriction A$ is bounded.

Then $(f\restriction]-\infty, c]{+\cdot}g\restriction[c, +\infty[)\restriction A$ is bounded. The theorem is a consequence of (23) and (35).

(37) Let us consider real numbers a, b, c, and functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant c \leqslant b$. Then $f\restriction[a, c[+\cdot g\restriction[c, b] = f\restriction[a, c]+\cdot g\restriction[c, b]$.
PROOF: Set $f_1 = f\restriction[a, c[+\cdot g\restriction[c, b]$. Set $f_2 = f\restriction[a, c]+\cdot g\restriction[c, b]$. For every object x such that $x \in \operatorname{dom} f_1$ holds $f_1(x) = f_2(x)$. $\square$

(38) Let us consider real numbers a, b, c, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \leqslant c$ and $h\restriction[a, c] = f\restriction[a, b]+\cdot g\restriction[b, c]$ and $f(b) = g(b)$. Then

(i) if $b \leqslant a$, then $h\restriction[a, c] = g\restriction[a, c]$, and

(ii) if $c \leqslant b$, then $h\restriction[a, c] = f\restriction[a, c]$.

PROOF: If $b \leqslant a$, then $h\restriction[a, c] = g\restriction[a, c]$. If $c \leqslant b$, then $h\restriction[a, c] = f\restriction[a, c]$. $\square$

(39) Let us consider a real number b, and functions f, g, h from $\mathbb{R}$ into $\mathbb{R}$. Suppose $h = f\restriction]-\infty, b[+\cdot g\restriction[b, +\infty[$ and $f(b) = g(b)$. Then

(i) if $b \leqslant \inf A$, then $h\restriction A = g\restriction A$, and

(ii) if $\sup A \leqslant b$, then $h\restriction A = f\restriction A$.

PROOF: If $b \leqslant \inf A$, then $h\restriction A = g\restriction A$ by [3, (4)]. If $\sup A \leqslant b$, then $h\restriction A = f\restriction A$ by [3, (4)]. $\square$

(40) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f = (\mathrm{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a-p}[+\cdot(\mathrm{AffineMap}(p, q))\restriction[\frac{q-b}{a-p}, +\infty[$ and $\frac{q-b}{a-p} \in A$.
Then $f\restriction A = (\mathrm{AffineMap}(a, b))\restriction[\inf A, \frac{q-b}{a-p}]+\cdot(\mathrm{AffineMap}(p, q))\restriction[\frac{q-b}{a-p}, \sup A]$.
PROOF: Set $F = (\mathrm{AffineMap}(a, b))\restriction[\inf A, \frac{q-b}{a-p}]+\cdot(\mathrm{AffineMap}(p, q))\restriction[\frac{q-b}{a-p}, \sup A]$. For every object x such that $x \in \operatorname{dom} F$ holds $F(x) = (f\restriction A)(x)$. $\square$

(41) Let us consider real numbers a, b. Then

(i) $(\mathrm{AffineMap}(a, b))\restriction A$ is bounded, and

(ii) $\mathrm{AffineMap}(a, b)$ is integrable on A.

Let us consider real numbers a, b, p, q and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(42) Suppose $a \neq p$ and $f = (\mathrm{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a-p}[+\cdot(\mathrm{AffineMap}(p, q))\restriction[\frac{q-b}{a-p}, +\infty[$. Then

(i) if $\frac{q-b}{a-p} \in A$, then $\int\limits_A f(x)dx = \int\limits_{[\inf A, \frac{q-b}{a-p}]} (\mathrm{AffineMap}(a, b))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\mathrm{AffineMap}(p, q))(x)dx$, and

(ii) if $\frac{q-b}{a-p} \leqslant \inf A$, then $\int\limits_A f(x)dx = \int\limits_A (\text{AffineMap}(p,q))(x)dx$, and

(iii) if $\frac{q-b}{a-p} \geqslant \sup A$, then $\int\limits_A f(x)dx = \int\limits_A (\text{AffineMap}(a,b))(x)dx$.

PROOF: $(\text{AffineMap}(a,b))(\frac{q-b}{a-p}) = (\text{AffineMap}(p,q))(\frac{q-b}{a-p})$. $\text{AffineMap}(a,b)$ is integrable on $[\inf A, \sup A]$ and $(\text{AffineMap}(a,b)){\restriction}[\inf A, \sup A]$ is bounded. $\text{AffineMap}(p,q)$ is integrable on $[\inf A, \sup A]$. $\text{AffineMap}(p,q){\restriction}[\inf A, \sup A]$ is bounded. f is integrable on $[\inf A, \sup A]$. If $\frac{q-b}{a-p} \in A$, then $\int\limits_A f(x)dx = \int\limits_{[\inf A, \frac{q-b}{a-p}]} (\text{AffineMap}(a,b))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\text{AffineMap}(p,q))(x)dx$. If $\frac{q-b}{a-p} \leqslant \inf A$, then $\int\limits_A f(x)dx = \int\limits_A (\text{AffineMap}(p,q))(x)dx$. If $\frac{q-b}{a-p} \geqslant \sup A$, then $\int\limits_A f(x)dx = \int\limits_A (\text{AffineMap}(a,b))(x)dx$. $\square$

(43) Suppose $a \neq p$ and $f{\restriction}A = \text{AffineMap}(a,b){\restriction}[\inf A, \frac{q-b}{a-p}] {+\cdot}\, \text{AffineMap}(p,q){\restriction}[\frac{q-b}{a-p}, \sup A]$ and $\frac{q-b}{a-p} \in A$. Then $\int\limits_A (\text{id}_{\mathbb{R}} \cdot f)(x)dx =$

$$\int\limits_{[\inf A, \frac{q-b}{a-p}]} (\text{id}_{\mathbb{R}} \cdot (\text{AffineMap}(a,b)))(x)dx + \int\limits_{[\frac{q-b}{a-p}, \sup A]} (\text{id}_{\mathbb{R}} \cdot (\text{AffineMap}(p,q)))(x)dx.$$

PROOF: $(\text{id}_{\mathbb{R}} \cdot f){\restriction}[\inf A, \sup A] = (\text{id}_{\mathbb{R}} \cdot (\text{AffineMap}(a,b))){\restriction}[\inf A, \frac{q-b}{a-p}] {+\cdot} (\text{id}_{\mathbb{R}} \cdot (\text{AffineMap}(p,q))){\restriction}[\frac{q-b}{a-p}, \sup A]$. Set $F = (\text{AffineMap}(a,b)){\restriction}]{-\infty}, \frac{q-b}{a-p}[{+\cdot}\, \text{AffineMap}(p,q){\restriction}[\frac{q-b}{a-p}, +\infty[$. $F{\restriction}[\inf A, \sup A]$ is integrable. $F{\restriction}[\inf A, \sup A] = f{\restriction}A$. f is integrable on $[\inf A, \sup A]$ and $f{\restriction}[\inf A, \sup A]$ is bounded. $\text{id}_{\mathbb{R}} \cdot f$ is integrable on $[\inf A, \sup A]$. $\square$

(44) Let us consider real numbers a, b. Then $\text{id}_{\mathbb{R}} \cdot \text{AffineMap}(a,b) = a \cdot \square^2 + b \cdot \square^1$.

PROOF: For every object x such that $x \in \mathbb{R}$ holds $\text{id}_{\mathbb{R}} \cdot \text{AffineMap}(a,b)(x) = a \cdot (\square^2 + b \cdot \square^1)(x)$. $\square$

(45) Let us consider real numbers a, b, c, d. Suppose $c \leqslant d$. Then $\int\limits_c^d (\text{id}_{\mathbb{R}} \cdot (\text{AffineMap}(a,b)))(x)dx = \frac{1}{3} \cdot a \cdot (d \cdot d \cdot d - c \cdot c \cdot c) + \frac{1}{2} \cdot b \cdot (d \cdot d - c \cdot c)$.

The theorem is a consequence of (44).

(46) Let us consider real numbers a, b. Then $\mathrm{AffineMap}(a,b) = a \cdot \square^1 + b \cdot \square^0$.
PROOF: For every object x such that $x \in \mathbb{R}$ holds $\mathrm{AffineMap}(a,b)(x) = (a \cdot \square^1 + b \cdot \square^0)(x)$. $\square$

(47) Let us consider real numbers a, b, c, d. Suppose $c \leqslant d$.
Then $\int\limits_c^d (\mathrm{AffineMap}(a,b))(x)dx = \frac{1}{2} \cdot a \cdot (d \cdot d - c \cdot c) + b \cdot (d - c)$. The theorem is a consequence of (46).

(48) Let us consider real numbers a, b, p, q, c, d, e, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a \neq p$ and $f \restriction A = \mathrm{AffineMap}(a,b) \restriction [\inf A, \frac{q-b}{a-p}] +\cdot\ \mathrm{AffineMap}(p,q) \restriction [\frac{q-b}{a-p}, \sup A]$ and $\frac{q-b}{a-p} \in A$. Then $\mathrm{centroid}(f, A) =$
$$\frac{\frac{1}{3} \cdot a \cdot ((\frac{q-b}{a-p})^{\mathbf{3}} - (\inf A)^{\mathbf{3}}) + \frac{1}{2} \cdot b \cdot ((\frac{q-b}{a-p})^{\mathbf{2}} - (\inf A)^{\mathbf{2}}) + \frac{1}{3} \cdot p \cdot ((\sup A)^{\mathbf{3}} - (\frac{q-b}{a-p})^{\mathbf{3}}) + \frac{1}{2} \cdot q \cdot ((\sup A)^{\mathbf{2}} - (\frac{q-b}{a-p})^{\mathbf{2}})}{\frac{1}{2} \cdot a \cdot ((\frac{q-b}{a-p})^{\mathbf{2}} - (\inf A)^{\mathbf{2}}) + b \cdot (\frac{q-b}{a-p} - \inf A) + \frac{1}{2} \cdot p \cdot ((\sup A)^{\mathbf{2}} - (\frac{q-b}{a-p})^{\mathbf{2}}) + q \cdot (\sup A - \frac{q-b}{a-p})}.$$
The theorem is a consequence of (18), (40), (42), (43), (45), and (47).

(49) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$.
Then $\max_+(f) = \max(\mathrm{AffineMap}(0,0), f)$.

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Accepted July 23, 2022