FORMALIZED MATHEMATICS
Vol. 30, No. 2, Pages 99–124, 2022
ISSN: 1426–2630, e-ISSN: 1898–9934
DOI: 10.2478/forma-2022-0009

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sciendo.com/journal/forma

Introduction to Graph Colorings

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Summary. In this article vertex, edge and total colorings of graphs are formalized in the Mizar system [4] and [1], based on the formalization of graphs in [5].

MSC: 68V20 05C15

Keywords: graph coloring; edge coloring; total coloring

MML identifier: GLCOLO00, version: 8.1.12 5.71.1431

Introduction

Graph coloring has a long history in mathematics and is introduced in almost every introductory book on graph theory (cf. [2], [6], [3]). In this article, the basic notions of vertex, edge and total colorings of graphs are formalized in sections 1, 2 and 3 respectively. These sections have the same basic structure.

At first the (not necessarily proper) coloring is defined as a function defined on the vertices or edges of a graph. The total coloring of a graph is defined as a pair of the other two.

The next definition is about proper colorings, i.e. that no two adjacent vertices or edges are colored the same. A proper total coloring also requires that vertices and edges who are incident with each other are not colored the same as well. In the context of this formalization, the vertex of a loop is considered adjacent to itself, but the edge of a loop is not considered adjacent to itself.

After that an attribute for proper colorability with a cardinal amount of colors is provided. It is important to note that the definition expresses how

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many colors are sufficient. Given that cardinalities can be infinite, an attribute indicating that only finitely many colors are needed is given as well.

In the last part of each section the chromatic number or index is introduced, indicating how many colors are at least necessary for a proper coloring.

1. Vertex Colorings

From now on E, V denote sets, G, G_1, G_2 denote graphs, c, c_1, c_2 denote cardinal numbers, and n denotes a natural number.

Let us consider G.

A vertex coloring of G is a many sorted set indexed by the vertices of G. One can check that every vertex coloring of G is non empty.

From now on f denotes a vertex coloring of G.

Now we state the proposition:

(1) Let us consider a function f'. Suppose $\operatorname{rng} f \subseteq \operatorname{dom} f'$. Then $f' \cdot f$ is a vertex coloring of G.

Let us consider G and f. Let f' be a many sorted set indexed by $\operatorname{rng} f$. One can check that the functor $f' \cdot f$ yields a vertex coloring of G. Now we state the propositions:

(2) Let us consider a vertex v of G, and an object x. Then $f {+}\cdot (v \dot{\longmapsto} x)$ is a vertex coloring of G.

(3) Let us consider a subgraph H of G. Then $f \restriction$(the vertices of H) is a vertex coloring of H.

(4) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, a vertex coloring f of G_2, and a function h. Suppose $\operatorname{dom} h = V \setminus$ (the vertices of G_2). Then $f {+}\cdot h$ is a vertex coloring of G_1.

(5) Let us consider objects v, e, x, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex coloring f of G_2. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then $f {+}\cdot (v \dot{\longmapsto} x)$ is a vertex coloring of G_1.

(6) Let us consider a vertex v of G_2, objects e, w, x, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex coloring f of G_2. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then $f {+}\cdot (w \dot{\longmapsto} x)$ is a vertex coloring of G_1.

(7) Let us consider objects v, x, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, and a vertex coloring f_2 of G_2. Suppose $v \notin$ the vertices of G_2. Then $f_2 {+}\cdot (v \dot{\longmapsto} x)$ is a vertex coloring of G_1.

Let us consider a partial graph mapping F from G_1 to G. Now we state the propositions:

(8) If $\mathrm{dom}(F_{\mathbb{V}})$ = the vertices of G_1, then $f \cdot (F_{\mathbb{V}})$ is a vertex coloring of G_1.

(9) If F is total, then $f \cdot (F_{\mathbb{V}})$ is a vertex coloring of G_1. The theorem is a consequence of (8).

Let us consider G and f. We say that f is proper if and only if

(Def. 1) for every vertices v, w of G such that v and w are adjacent holds $f(v) \neq f(w)$.

Now we state the propositions:

(10) f is proper if and only if for every objects e, v, w such that e joins v and w in G holds $f(v) \neq f(w)$.

(11) f is proper if and only if for every objects e, v, w such that e joins v to w in G holds $f(v) \neq f(w)$. The theorem is a consequence of (10).

(12) Let us consider a one-to-one function f', and a vertex coloring f_2 of G. Suppose $f_2 = f' \cdot f$ and f is proper and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f'$. Then f_2 is proper. The theorem is a consequence of (10).

(13) Let us consider a one-to-one many sorted set f' indexed by $\mathrm{rng}\, f$. If f is proper, then $f' \cdot f$ is proper. The theorem is a consequence of (12).

(14) If there exists f such that f is proper, then G is loopless. The theorem is a consequence of (10).

Let G be a non loopless graph. Observe that every vertex coloring of G is non proper.

Let G be a loopless graph. Let us observe that every vertex coloring of G which is one-to-one is also proper and there exists a vertex coloring of G which is one-to-one and proper.

Now we state the propositions:

(15) Let us consider a subgraph H of G, and a vertex coloring f' of H. Suppose $f' = f{\restriction}$(the vertices of H) and f is proper. Then f' is proper. The theorem is a consequence of (10).

(16) Let us consider a vertex coloring f_1 of G_1, and a vertex coloring f_2 of G_2. Suppose $G_1 \approx G_2$ and $f_1 = f_2$ and f_1 is proper. Then f_2 is proper. The theorem is a consequence of (10).

(17) Let us consider a vertex coloring f_1 of G_1, a vertex coloring f_2 of G_2, a vertex v of G_1, and an object x. Suppose $G_1 \approx G_2$ and $f_2 = f_1 {+\cdot} (v \mapsto x)$ and $x \notin \mathrm{rng}\, f_1$ and f_1 is proper. Then f_2 is proper. The theorem is a consequence of (10).

(18) Let us consider a graph G_2 given by reversing directions of the edges E of G_1, a vertex coloring f_1 of G_1, and a vertex coloring f_2 of G_2. If $f_1 = f_2$,

then f_1 is proper iff f_2 is proper.

(19) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, a vertex coloring f_1 of G_1, a vertex coloring f_2 of G_2, and a function h. Suppose $\operatorname{dom} h = V \setminus$ (the vertices of G_2) and $f_1 = f_2 {+\cdot} h$ and f_2 is proper. Then f_1 is proper. The theorem is a consequence of (10).

(20) Let us consider vertices v, w of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v and w, a vertex coloring f_1 of G_1, and a vertex coloring f_2 of G_2. Suppose $f_1 = f_2$ and v and w are adjacent and f_2 is proper. Then f_1 is proper. The theorem is a consequence of (10) and (16).

(21) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by e between vertices v and w, a vertex coloring f_1 of G_1, a vertex coloring f_2 of G_2, and an object x. Suppose $f_1 = f_2 {+\cdot} (v \mathrel{\dot{\longmapsto}} x)$ and $v \neq w$ and $x \notin \operatorname{rng} f_2$ and f_2 is proper. Then f_1 is proper. The theorem is a consequence of (10) and (17).

(22) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, a vertex coloring f_1 of G_1, a vertex coloring f_2 of G_2, and an object x. Suppose $f_1 = f_2 {+\cdot} (w \mathrel{\dot{\longmapsto}} x)$ and $v \neq w$ and $x \notin \operatorname{rng} f_2$ and f_2 is proper. Then f_1 is proper. The theorem is a consequence of (21), (18), and (17).

Let us consider objects v, e, w, a supergraph G_1 of G_2 extended by v, w and e between them, a vertex coloring f_1 of G_1, a vertex coloring f_2 of G_2, and an object x. Now we state the propositions:

(23) Suppose $v \notin$ the vertices of G_2 and $f_1 = f_2 {+\cdot} (v \mathrel{\dot{\longmapsto}} x)$ and $x \neq f_2(w)$. Then if f_2 is proper, then f_1 is proper. The theorem is a consequence of (11).

(24) Suppose $w \notin$ the vertices of G_2 and $f_1 = f_2 {+\cdot} (w \mathrel{\dot{\longmapsto}} x)$ and $x \neq f_2(v)$. Then if f_2 is proper, then f_1 is proper. The theorem is a consequence of (23) and (18).

(25) Let us consider objects v, x, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, a vertex coloring f_1 of G_1, and a vertex coloring f_2 of G_2. Suppose $v \notin$ the vertices of G_2 and $f_1 = f_2 {+\cdot} (v \mathrel{\dot{\longmapsto}} x)$ and $x \notin \operatorname{rng} f_2$. If f_2 is proper, then f_1 is proper. The theorem is a consequence of (10).

(26) Let us consider a partial graph mapping F from G_1 to G, and a vertex coloring f' of G_1. Suppose F is total and $f' = f \cdot (F_{\mathbb{V}})$ and f is proper. Then f' is proper. The theorem is a consequence of (10).

Let us consider c and G. We say that G is c-vertex-colorable if and only if

(Def. 2) there exists a vertex coloring f of G such that f is proper and $\overline{\overline{\mathrm{rng}\, f}} \subseteq c$.

Now we state the propositions:

(27) If $c_1 \subseteq c_2$ and G is c_1-vertex-colorable, then G is c_2-vertex-colorable.

(28) If there exists c such that G is c-vertex-colorable, then G is loopless.

Let us consider c. Note that every graph which is c-vertex-colorable is also loopless and every graph which is loopless and c-vertex is also c-vertex-colorable and every graph is non 0-vertex-colorable.

Now we state the propositions:

(29) If G is loopless, then G is (G.order())-vertex-colorable.

(30) G is edgeless if and only if G is 1-vertex-colorable. The theorem is a consequence of (10).

Let c be a non zero cardinal number. One can verify that there exists a graph which is c-vertex-colorable.

Now we state the proposition:

(31) Let us consider a subgraph H of G. If G is c-vertex-colorable, then H is c-vertex-colorable. The theorem is a consequence of (3) and (15).

One can verify that every graph which is edgeless is also 1-vertex-colorable and every graph which is 1-vertex-colorable is also edgeless.

Let c be a non zero cardinal number and G be a c-vertex-colorable graph. Let us observe that every subgraph of G is c-vertex-colorable.

Now we state the propositions:

(32) If $G_1 \approx G_2$ and G_1 is c-vertex-colorable, then G_2 is c-vertex-colorable. The theorem is a consequence of (16).

(33) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable.

Let c be a non zero cardinal number and G_1 be a c-vertex-colorable graph. Let us consider E. One can verify that every graph given by reversing directions of the edges E of G_1 is c-vertex-colorable.

Now we state the proposition:

(34) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable. The theorem is a consequence of (31), (4), and (19).

Let c be a non zero cardinal number and G_2 be a c-vertex-colorable graph. Let us consider V. One can verify that every supergraph of G_2 extended by the vertices from V is c-vertex-colorable.

Now we state the propositions:

(35) Let us consider vertices v, w of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose v and w are adjacent.

Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable. The theorem is a consequence of (31) and (20).

(36) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose $v \neq w$ and G_2 is c-vertex-colorable. Then G_1 is $(c+1)$-vertex-colorable. The theorem is a consequence of (22), (32), and (27).

(37) Let us consider a non edgeless graph G_2, objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable. The theorem is a consequence of (31), (33), and (32).

(38) Let us consider an edgeless graph G_2, and objects v, e, w. Then every supergraph of G_2 extended by v, w and e between them is 2-vertex-colorable. The theorem is a consequence of (33), (32), and (27).

(39) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. If G_2 is c-vertex-colorable, then G_1 is $(c+1)$-vertex-colorable. The theorem is a consequence of (7), (25), (32), and (27).

(40) Let us consider a subgraph G_2 of G_1 with parallel edges removed. Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable. The theorem is a consequence of (31).

Let c be a non zero cardinal number and G_1 be a c-vertex-colorable graph. Note that every subgraph of G_1 with parallel edges removed is c-vertex-colorable.

Now we state the proposition:

(41) Let us consider a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1 is c-vertex-colorable if and only if G_2 is c-vertex-colorable. The theorem is a consequence of (31) and (40).

Let c be a non zero cardinal number and G_1 be a c-vertex-colorable graph. One can check that every subgraph of G_1 with directed-parallel edges removed is c-vertex-colorable.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(42) If F is weak subgraph embedding and G_2 is c-vertex-colorable, then G_1 is c-vertex-colorable. The theorem is a consequence of (9) and (26).

(43) If F is isomorphism, then G_1 is c-vertex-colorable iff G_2 is c-vertex-colorable. The theorem is a consequence of (42).

Let c be a non zero cardinal number and G be a c-vertex-colorable graph. Let us note that every graph which is G-isomorphic is also c-vertex-colorable.

Let us consider G. We say that G is finitely vertex-colorable if and only if

(Def. 3) there exists n such that G is n-vertex-colorable.

One can verify that every graph which is finitely vertex-colorable is also loopless and every graph which is vertex-finite and loopless is also finitely vertex-colorable and every graph which is edgeless is also finitely vertex-colorable.

Let us consider n. Let us note that every graph which is n-vertex-colorable is also finitely vertex-colorable and there exists a graph which is finitely vertex-colorable and there exists a graph which is non finitely vertex-colorable.

Let G be a finitely vertex-colorable graph. Observe that every subgraph of G is finitely vertex-colorable.

Let G be a non finitely vertex-colorable graph. One can verify that every supergraph of G is non finitely vertex-colorable.

Now we state the propositions:

(44) If $G_1 \approx G_2$ and G_1 is finitely vertex-colorable, then G_2 is finitely vertex-colorable. The theorem is a consequence of (32).

(45) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable.

Let G_1 be a finitely vertex-colorable graph. Let us consider E. Observe that every graph given by reversing directions of the edges E of G_1 is finitely vertex-colorable.

Let G_1 be a non finitely vertex-colorable graph. Note that every graph given by reversing directions of the edges E of G_1 is non finitely vertex-colorable.

Now we state the proposition:

(46) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (34).

Let G_2 be a finitely vertex-colorable graph. Let us consider V. One can verify that every supergraph of G_2 extended by the vertices from V is finitely vertex-colorable.

Now we state the propositions:

(47) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose $v \neq w$. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (36).

(48) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (37) and (38).

Let G_2 be a finitely vertex-colorable graph and v, e, w be objects. Observe that every supergraph of G_2 extended by v, w and e between them is finitely vertex-colorable.

Now we state the proposition:

(49) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (39).

Let G_2 be a finitely vertex-colorable graph and v be an object. Let us consider V. Let us note that every supergraph of G_2 extended by vertex v and edges between v and V of G_2 is finitely vertex-colorable.

Now we state the proposition:

(50) Let us consider a subgraph G_2 of G_1 with parallel edges removed. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (40).

Let G_1 be a non finitely vertex-colorable graph. One can verify that every subgraph of G_1 with parallel edges removed is non finitely vertex-colorable.

Now we state the proposition:

(51) Let us consider a subgraph G_2 of G_1 with directed-parallel edges removed. Then G_1 is finitely vertex-colorable if and only if G_2 is finitely vertex-colorable. The theorem is a consequence of (41).

Let G_1 be a non finitely vertex-colorable graph. One can verify that every subgraph of G_1 with directed-parallel edges removed is non finitely vertex-colorable.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(52) If F is weak subgraph embedding and G_2 is finitely vertex-colorable, then G_1 is finitely vertex-colorable. The theorem is a consequence of (42).

(53) If F is isomorphism, then G_1 is finitely vertex-colorable iff G_2 is finitely vertex-colorable. The theorem is a consequence of (52).

Let G be a finitely vertex-colorable graph. Observe that every graph which is G-isomorphic is also finitely vertex-colorable.

Let G be a graph. The functor $\chi(G)$ yielding a cardinal number is defined by the term

(Def. 4) $\bigcap\{c$, where c is a cardinal subset of G.order() : G is c-vertex-colorable$\}$.

Now we state the propositions:

(54) If G is loopless, then G is $\chi(G)$-vertex-colorable. The theorem is a consequence of (29).

(55) G is not loopless if and only if $\chi(G) = 0$. The theorem is a consequence of (29).

Let G be a loopless graph. One can verify that $\chi(G)$ is non zero.

Let G be a non loopless graph. Let us observe that $\chi(G)$ is zero.

Now we state the propositions:

(56) $\chi(G) \subseteq G.\mathrm{order}()$. The theorem is a consequence of (29).

(57) If G is c-vertex-colorable, then $\chi(G) \subseteq c$. The theorem is a consequence of (56).

(58) If G is c-vertex-colorable and for every cardinal number d such that G is d-vertex-colorable holds $c \subseteq d$, then $\chi(G) = c$. The theorem is a consequence of (57) and (29).

Let G be a finitely vertex-colorable graph. Note that $\chi(G)$ is natural.

Let us note that the functor $\chi(G)$ yields a natural number. Now we state the propositions:

(59) Let us consider a loopless graph G. Then $1 \subseteq \chi(G)$.

(60) G is edgeless if and only if $\chi(G) = 1$. The theorem is a consequence of (57), (59), and (54).

(61) Let us consider a loopless, non edgeless graph G. Then $2 \subseteq \chi(G)$. The theorem is a consequence of (60).

(62) Let us consider a loopless graph G. If G is complete, then $\chi(G) = G.\mathrm{order}()$. The theorem is a consequence of (29) and (56).

(63) Let us consider a loopless graph G, and a subgraph H of G. Then $\chi(H) \subseteq \chi(G)$. The theorem is a consequence of (54) and (57).

(64) If $G_1 \approx G_2$, then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (32).

(65) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (33).

(66) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (54), (34), (57), and (58).

(67) Let us consider a non edgeless graph G_2, objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (54), (37), (57), and (58).

(68) Let us consider an edgeless graph G_2, a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $w \notin$ the vertices of G_2. Then $\chi(G_1) = 2$. The theorem is a consequence of (38) and (58).

(69) Let us consider an edgeless graph G_2, objects v, e, a vertex w of G_2, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose

$v \notin$ the vertices of G_2. Then $\chi(G_1) = 2$. The theorem is a consequence of (38) and (58).

(70) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then $\chi(G_1) \subseteq \chi(G_2)+1$. The theorem is a consequence of (54), (39), and (57).

(71) Let us consider a loopless graph G_2, an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and the vertices of G_2. Suppose $v \notin$ the vertices of G_2. Then $\chi(G_1) = \chi(G_2) + 1$. The theorem is a consequence of (70), (63), (54), (3), (15), and (57).

(72) Let us consider a subgraph G_2 of G_1 with parallel edges removed. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (40), (54), (57), and (58).

(73) Let us consider a subgraph G_2 of G_1 with directed-parallel edges removed. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (41), (54), (57), and (58).

(74) Let us consider a graph G_1, a loopless graph G_2, and a partial graph mapping F from G_1 to G_2. If F is weak subgraph embedding, then $\chi(G_1) \subseteq \chi(G_2)$. The theorem is a consequence of (42), (54), and (57).

(75) Let us consider a partial graph mapping F from G_1 to G_2. If F is isomorphism, then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (54), (43), (57), and (58).

(76) Let us consider a G_1-isomorphic graph G_2. Then $\chi(G_1) = \chi(G_2)$. The theorem is a consequence of (75).

2. Edge Colorings

Let us consider G.

An edge coloring of G is a many sorted set indexed by the edges of G. In the sequel g denotes an edge coloring of G.

Now we state the proposition:

(77) Let us consider a function g'. Suppose $\operatorname{rng} g \subseteq \operatorname{dom} g'$. Then $g' \cdot g$ is an edge coloring of G.

Let us consider G and g. Let g' be a many sorted set indexed by $\operatorname{rng} g$. Note that the functor $g' \cdot g$ yields an edge coloring of G. Now we state the propositions:

(78) Let us consider a subgraph H of G. Then $g{\restriction}$(the edges of H) is an edge coloring of H.

(79) Let us consider an object e, vertices v, w of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, an edge coloring g of G_2, and

an object x. Suppose $e \notin$ the edges of G_2. Then $g{+}\cdot(e \dot{\longmapsto} x)$ is an edge coloring of G_1.

(80) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, an edge coloring g of G_2, and an object x. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then $g{+}\cdot(e \dot{\longmapsto} x)$ is an edge coloring of G_1.

(81) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, an edge coloring g of G_2, and an object x. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then $g{+}\cdot(e \dot{\longmapsto} x)$ is an edge coloring of G_1.

(82) Let us consider an object v, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, an edge coloring g_2 of G_2, and a function h. Suppose $v \notin$ the vertices of G_2 and $\operatorname{dom} h = G_1.\text{edgesBetween}(V, \{v\})$. Then $g_2{+}\cdot h$ is an edge coloring of G_1.

Let us consider a partial graph mapping F from G_1 to G. Now we state the propositions:

(83) If $\operatorname{dom}(F_{\mathbb{E}}) =$ the edges of G_1, then $g \cdot (F_{\mathbb{E}})$ is an edge coloring of G_1.

(84) If F is total, then $g \cdot (F_{\mathbb{E}})$ is an edge coloring of G_1. The theorem is a consequence of (83).

Let us consider G and g. We say that g is proper if and only if

(Def. 5) for every vertex v of G, $g{\restriction}v.\text{edgesInOut}()$ is one-to-one.

Now we state the propositions:

(85) g is proper if and only if for every vertex v of G and for every objects e_1, e_2 such that $e_1, e_2 \in v.\text{edgesInOut}()$ and $g(e_1) = g(e_2)$ holds $e_1 = e_2$.

(86) g is proper if and only if for every objects e_1, e_2, v, w_1, w_2 such that e_1 joins v and w_1 in G and e_2 joins v and w_2 in G and $g(e_1) = g(e_2)$ holds $e_1 = e_2$. The theorem is a consequence of (85).

(87) Let us consider a one-to-one function g', and an edge coloring g_2 of G. If $g_2 = g' \cdot g$ and g is proper, then g_2 is proper.

(88) Let us consider a one-to-one many sorted set g' indexed by $\operatorname{rng} g$. If g is proper, then $g' \cdot g$ is proper.

Let us consider G. One can verify that every edge coloring of G which is one-to-one is also proper and there exists an edge coloring of G which is one-to-one and proper.

Now we state the propositions:

(89) Let us consider a subgraph H of G, and an edge coloring g' of H. Suppose $g' = g{\restriction}(\text{the edges of } H)$ and g is proper. Then g' is proper. The theorem is a consequence of (85).

(90) Let us consider an edge coloring g_1 of G_1, and an edge coloring g_2 of G_2. Suppose $G_1 \approx G_2$ and $g_1 = g_2$ and g_1 is proper. Then g_2 is proper.

(91) Let us consider a graph G_2 given by reversing directions of the edges E of G_1, an edge coloring g_1 of G_1, and an edge coloring g_2 of G_2. If $g_1 = g_2$, then g_1 is proper iff g_2 is proper.

(92) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, an edge coloring g_1 of G_1, and an edge coloring g_2 of G_2. If $g_1 = g_2$, then if g_2 is proper, then g_1 is proper.

(93) Let us consider objects v, e, w, a supergraph G_1 of G_2 extended by e between vertices v and w, an edge coloring g_1 of G_1, an edge coloring g_2 of G_2, and an object x. Suppose $g_1 = g_2 {+\cdot} (e \mathrel{\dot{\longmapsto}} x)$ and $e \notin$ the edges of G_2 and $x \notin \operatorname{rng} g_2$. If g_2 is proper, then g_1 is proper.

(94) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, an edge coloring g_1 of G_1, an edge coloring g_2 of G_2, and an object x. Suppose $g_1 = g_2 {+\cdot} (e \mathrel{\dot{\longmapsto}} x)$ and $x \notin \operatorname{rng} g_2$ and $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. If g_2 is proper, then g_1 is proper. The theorem is a consequence of (92) and (93).

(95) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, an edge coloring g_1 of G_1, an edge coloring g_2 of G_2, and an object x. Suppose $g_1 = g_2 {+\cdot} (e \mathrel{\dot{\longmapsto}} x)$ and $x \notin \operatorname{rng} g_2$ and $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. If g_2 is proper, then g_1 is proper. The theorem is a consequence of (92) and (93).

(96) Let us consider an object v, a subset V of the vertices of G_2, a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2, an edge coloring g_2 of G_2, an edge coloring g_1 of G_1, and sets X, E. Suppose $E = G_1.\mathrm{edgesBetween}(V, \{v\})$ and $\operatorname{rng} g_2 \subseteq X$ and $g_1 = g_2 {+\cdot} \langle E \longmapsto X, \mathrm{id}_E \rangle$ and $v \notin$ the vertices of G_2 and g_2 is proper. Then g_1 is proper. The theorem is a consequence of (85) and (86).

Let us consider a partial graph mapping F from G_1 to G and an edge coloring g' of G_1. Now we state the propositions:

(97) Suppose $\operatorname{dom}(F_{\mathbb{E}}) =$ the edges of G_1 and $F_{\mathbb{E}}$ is one-to-one and $g' = g \cdot (F_{\mathbb{E}})$ and g is proper. Then g' is proper. The theorem is a consequence of (85).

(98) If F is weak subgraph embedding and $g' = g \cdot (F_{\mathbb{E}})$ and g is proper, then g' is proper. The theorem is a consequence of (97).

Let us consider c and G. We say that G is c-edge-colorable if and only if

(Def. 6) there exists a proper edge coloring g of G such that $\overline{\overline{\operatorname{rng} g}} \subseteq c$.

Now we state the propositions:

(99) If $c_1 \subseteq c_2$ and G is c_1-edge-colorable, then G is c_2-edge-colorable.

(100) G is (G.size())-edge-colorable.

(101) G is edgeless if and only if G is 0-edge-colorable. The theorem is a consequence of (100).

Let us observe that every graph which is edgeless is also 0-edge-colorable and every graph which is 0-edge-colorable is also edgeless.

Let us consider c. Note that every graph which is c-edge is also c-edge-colorable and there exists a graph which is c-edge-colorable.

Now we state the proposition:

(102) Let us consider a subgraph H of G. If G is c-edge-colorable, then H is c-edge-colorable. The theorem is a consequence of (78) and (89).

Let us consider c. Let G be a c-edge-colorable graph. Note that every subgraph of G is c-edge-colorable.

Now we state the propositions:

(103) If $G_1 \approx G_2$ and G_1 is c-edge-colorable, then G_2 is c-edge-colorable. The theorem is a consequence of (90).

(104) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is c-edge-colorable if and only if G_2 is c-edge-colorable.

Let us consider c. Let G_1 be a c-edge-colorable graph. Let us consider E. Let us note that every graph given by reversing directions of the edges E of G_1 is c-edge-colorable.

Now we state the proposition:

(105) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is c-edge-colorable if and only if G_2 is c-edge-colorable. The theorem is a consequence of (92).

Let us consider c. Let G_2 be a c-edge-colorable graph. Let us consider V. Let us note that every supergraph of G_2 extended by the vertices from V is c-edge-colorable.

Let us consider a c-edge-colorable graph G_2 and objects v, e, w. Now we state the propositions:

(106) Every supergraph of G_2 extended by e between vertices v and w is $(c+1)$-edge-colorable. The theorem is a consequence of (79), (93), (103), and (99).

(107) Every supergraph of G_2 extended by v, w and e between them is $(c+1)$-edge-colorable. The theorem is a consequence of (106), (103), and (99).

Now we state the proposition:

(108) Let us consider an edgeless graph G_2, and objects v, e, w. Then every supergraph of G_2 extended by v, w and e between them is 1-edge-colorable. The theorem is a consequence of (104) and (99).

Let us consider c. Let G_2 be a c-edge-colorable graph and v, e, w be objects. Note that every supergraph of G_2 extended by e between vertices v and w is $(c+1)$-edge-colorable and every supergraph of G_2 extended by v, w and e between them is $(c+1)$-edge-colorable.

Now we state the proposition:

(109) Let us consider a c-edge-colorable graph G_2, and an object v. Then every supergraph of G_2 extended by vertex v and edges between v and V of G_2 is $(c+\overline{\overline{V}})$-edge-colorable. The theorem is a consequence of (82), (96), (103), and (99).

Let us consider c. Let G_2 be a c-edge-colorable graph and v be an object. Let us consider V. One can verify that every supergraph of G_2 extended by vertex v and edges between v and V of G_2 is $(c+\overline{\overline{V}})$-edge-colorable.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(110) If F is weak subgraph embedding and G_2 is c-edge-colorable, then G_1 is c-edge-colorable. The theorem is a consequence of (84) and (98).

(111) If F is isomorphism, then G_1 is c-edge-colorable iff G_2 is c-edge-colorable. The theorem is a consequence of (110).

Let us consider c. Let G be a c-edge-colorable graph. Note that every graph which is G-isomorphic is also c-edge-colorable.

Let us consider G. We say that G is finitely edge-colorable if and only if

(Def. 7) there exists n such that G is n-edge-colorable.

Let us observe that every graph which is edge-finite is also finitely edge-colorable and every graph which is edgeless is also finitely edge-colorable and every graph which is finitely edge-colorable is also locally-finite.

Let us consider n. One can check that every graph which is n-edge-colorable is also finitely edge-colorable and there exists a graph which is finitely edge-colorable and there exists a graph which is non finitely edge-colorable.

Let G be a finitely edge-colorable graph. Note that every subgraph of G is finitely edge-colorable.

Now we state the propositions:

(112) If $G_1 \approx G_2$ and G_1 is finitely edge-colorable, then G_2 is finitely edge-colorable. The theorem is a consequence of (103).

(113) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is finitely edge-colorable if and only if G_2 is finitely edge-colorable.

Let G_1 be a finitely edge-colorable graph. Let us consider E. One can verify that every graph given by reversing directions of the edges E of G_1 is finitely edge-colorable.

Let G_1 be a non finitely edge-colorable graph. Observe that every graph given by reversing directions of the edges E of G_1 is non finitely edge-colorable.

Now we state the proposition:

(114) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is finitely edge-colorable if and only if G_2 is finitely edge-colorable. The theorem is a consequence of (105).

Let G_2 be a finitely edge-colorable graph. Let us consider V. One can verify that every supergraph of G_2 extended by the vertices from V is finitely edge-colorable.

Let G_2 be a non finitely edge-colorable graph. Observe that every supergraph of G_2 extended by the vertices from V is non finitely edge-colorable.

Now we state the proposition:

(115) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Then G_1 is finitely edge-colorable if and only if G_2 is finitely edge-colorable. The theorem is a consequence of (107).

Let G_2 be a finitely edge-colorable graph and v, e, w be objects. Note that every supergraph of G_2 extended by e between vertices v and w is finitely edge-colorable.

Let G_2 be a non finitely edge-colorable graph. One can verify that every supergraph of G_2 extended by e between vertices v and w is non finitely edge-colorable.

Now we state the proposition:

(116) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then G_1 is finitely edge-colorable if and only if G_2 is finitely edge-colorable.

Let G_2 be a finitely edge-colorable graph and v, e, w be objects. Observe that every supergraph of G_2 extended by v, w and e between them is finitely edge-colorable.

Let G_2 be a non finitely edge-colorable graph. Note that every supergraph of G_2 extended by v, w and e between them is non finitely edge-colorable.

Now we state the proposition:

(117) Let us consider an object v, a finite set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then G_1 is finitely edge-colorable if and only if G_2 is finitely edge-colorable.

Let G_2 be a finitely edge-colorable graph, v be an object, and V be a finite set. Let us observe that every supergraph of G_2 extended by vertex v and edges between v and V of G_2 is finitely edge-colorable.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(118) If F is weak subgraph embedding and G_2 is finitely edge-colorable, then G_1 is finitely edge-colorable. The theorem is a consequence of (110).

(119) If F is isomorphism, then G_1 is finitely edge-colorable iff G_2 is finitely edge-colorable. The theorem is a consequence of (118).

Let G be a finitely edge-colorable graph. One can verify that every graph which is G-isomorphic is also finitely edge-colorable.

Let us consider G. The functor $\chi'(G)$ yielding a cardinal number is defined by the term

(Def. 8) $\bigcap\{c$, where c is a cardinal subset of G.size() : G is c-edge-colorable$\}$.

Now we state the propositions:

(120) $\chi'(G) \subseteq G$.size(). The theorem is a consequence of (100).

(121) G is edgeless if and only if $\chi'(G) = 0$. The theorem is a consequence of (120).

Let G be an edgeless graph. One can check that $\chi'(G)$ is zero.

Let G be a non edgeless graph. One can check that $\chi'(G)$ is non zero.

Now we state the proposition:

(122) G is c-edge-colorable and for every cardinal number d such that G is d-edge-colorable holds $c \subseteq d$ if and only if $\chi'(G) = c$. The theorem is a consequence of (100).

Let G be a finitely edge-colorable graph. Let us observe that $\chi'(G)$ is natural.

Let us observe that the functor $\chi'(G)$ yields a natural number. Now we state the propositions:

(123) Let us consider a loopless graph G. Then $\bar{\Delta}(G) \subseteq \chi'(G)$.

(124) If $G_1 \approx G_2$, then $\chi'(G_1) = \chi'(G_2)$. The theorem is a consequence of (103) and (122).

(125) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then $\chi'(G_1) = \chi'(G_2)$. The theorem is a consequence of (104) and (122).

(126) Let us consider a subgraph H of G. Then $\chi'(H) \subseteq \chi'(G)$.

(127) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then $\chi'(G_1) = \chi'(G_2)$. The theorem is a consequence of (105) and (122).

(128) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Then $\chi'(G_1) \subseteq \chi'(G_2) + 1$. The theorem is a consequence of (106).

(129) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then $\chi'(G_1) \subseteq \chi'(G_2) + 1$. The theorem is a consequence of (107).

(130) Let us consider an edgeless graph G_2, a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $w \notin$ the vertices of G_2. Then $\chi'(G_1) = 1$. The theorem is a consequence of (122).

(131) Let us consider an edgeless graph G_2, objects v, e, a vertex w of G_2, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $v \notin$ the vertices of G_2. Then $\chi'(G_1) = 1$. The theorem is a consequence of (130) and (125).

(132) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then $\chi'(G_1) \subseteq \chi'(G_2) + \overline{\overline{V}}$.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(133) If F is weak subgraph embedding, then $\chi'(G_1) \subseteq \chi'(G_2)$. The theorem is a consequence of (110).

(134) If F is isomorphism, then $\chi'(G_1) = \chi'(G_2)$. The theorem is a consequence of (133).

(135) Let us consider a G_1-isomorphic graph G_2. Then $\chi'(G_1) = \chi'(G_2)$. The theorem is a consequence of (134).

(136) If G is trivial, then $\chi'(G) = G$.size(). The theorem is a consequence of (100) and (122).

3. Total Colorings

Let us consider G.

A total coloring of G is an object defined by

(Def. 9) there exists a vertex coloring f of G and there exists an edge coloring g of G such that $it = \langle f, g \rangle$.

Note that every total coloring of G is pair.

From now on t denotes a total coloring of G.

Let us consider G and t. We introduce the notation $t_{\mathbb{V}}$ as a synonym of $(t)_{\mathbf{1}}$ and $t_{\mathbb{E}}$ as a synonym of $(t)_{\mathbf{2}}$.

One can check that $\langle t_{\mathbb{V}}, t_{\mathbb{E}} \rangle$ reduces to t.

One can verify that the functor $t_{\mathbb{V}}$ yields a vertex coloring of G. Let us observe that the functor $t_{\mathbb{E}}$ yields an edge coloring of G. Let us consider f and g. Note that the functor $\langle f, g \rangle$ yields a total coloring of G. Now we state the propositions:

(137) If G is edgeless, then $\langle f, \emptyset \rangle$ is a total coloring of G.

(138) Let us consider a subgraph H of G. Then $\langle t_{\mathbb{V}}\restriction$(the vertices of H), $t_{\mathbb{E}}\restriction$(the edges of H)$\rangle$ is a total coloring of H. The theorem is a consequence of (3) and (78).

(139) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, a total coloring t of G_2, and a function h. Suppose $\operatorname{dom} h = V \setminus$ (the vertices of G_2). Then $\langle t_{\mathbb{V}}{+\cdot}h, t_{\mathbb{E}}\rangle$ is a total coloring of G_1. The theorem is a consequence of (4).

(140) Let us consider objects v, x, a supergraph G_1 of G_2 extended by v, and a total coloring t of G_2. Then $\langle t_{\mathbb{V}}{+\cdot}(v \dot{\longmapsto} x), t_{\mathbb{E}}\rangle$ is a total coloring of G_1.

(141) Let us consider an object e, vertices v, w of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, a total coloring t of G_2, and an object y. Suppose $e \notin$ the edges of G_2. Then $\langle t_{\mathbb{V}}, t_{\mathbb{E}}{+\cdot}(e \dot{\longmapsto} y)\rangle$ is a total coloring of G_1.

(142) Let us consider an object e, vertices v, w, u of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, a total coloring t of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2. Then $\langle t_{\mathbb{V}}{+\cdot}(u \dot{\longmapsto} x), t_{\mathbb{E}}{+\cdot}(e \dot{\longmapsto} y)\rangle$ is a total coloring of G_1. The theorem is a consequence of (141).

(143) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, a total coloring t of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then $\langle t_{\mathbb{V}}{+\cdot}(v \dot{\longmapsto} x), t_{\mathbb{E}}{+\cdot}(e \dot{\longmapsto} y)\rangle$ is a total coloring of G_1. The theorem is a consequence of (140) and (141).

(144) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, a total coloring t of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then $\langle t_{\mathbb{V}}{+\cdot}(w \dot{\longmapsto} x), t_{\mathbb{E}}{+\cdot}(e \dot{\longmapsto} y)\rangle$ is a total coloring of G_1. The theorem is a consequence of (140) and (141).

(145) Let us consider a partial graph mapping F from G_1 to G. Suppose F is total. Then $\langle (t_{\mathbb{V}}) \cdot (F_{\mathbb{V}}), (t_{\mathbb{E}}) \cdot (F_{\mathbb{E}})\rangle$ is a total coloring of G_1. The theorem is a consequence of (9) and (84).

Let us consider G and t. We say that t is proper if and only if

(Def. 10) $t_{\mathbb{V}}$ is proper and $t_{\mathbb{E}}$ is proper and for every vertex v of G, $(t_{\mathbb{V}})(v) \notin (t_{\mathbb{E}})^{\circ}(v.\text{edgesInOut}())$.

Now we state the propositions:

(146) t is proper if and only if $t_{\mathbb{V}}$ is proper and $t_{\mathbb{E}}$ is proper and for every objects e, v, w such that e joins v and w in G holds $(t_{\mathbb{V}})(v) \neq (t_{\mathbb{E}})(e)$.

(147) If $t_{\mathbb{V}}$ is proper and $t_{\mathbb{E}}$ is proper and $\operatorname{rng} t_{\mathbb{V}}$ misses $\operatorname{rng} t_{\mathbb{E}}$, then t is proper. The theorem is a consequence of (146).

(148) t is proper if and only if for every objects e_1, e_2, v, w_1, w_2 such that e_1 joins v and w_1 in G and e_2 joins v and w_2 in G holds $(t_{\mathbb{V}})(v) \neq (t_{\mathbb{V}})(w_1)$ and $(t_{\mathbb{V}})(v) \neq (t_{\mathbb{E}})(e_1)$ and if $e_1 \neq e_2$, then $(t_{\mathbb{E}})(e_1) \neq (t_{\mathbb{E}})(e_2)$. The theorem is a consequence of (10), (86), and (146).

(149) Suppose g is proper. Then there exists a proper edge coloring g' of G such that

(i) $\operatorname{rng} f$ misses $\operatorname{rng} g'$, and

(ii) $\overline{\overline{\operatorname{rng} g}} = \overline{\overline{\operatorname{rng} g'}}$.

The theorem is a consequence of (77) and (87).

(150) Suppose f is proper. Then there exists a vertex coloring f' of G such that

(i) f' is proper, and

(ii) $\operatorname{rng} f'$ misses $\operatorname{rng} g$, and

(iii) $\overline{\overline{\operatorname{rng} f}} = \overline{\overline{\operatorname{rng} f'}}$.

The theorem is a consequence of (1) and (12).

Let G be a loopless graph. Observe that there exists a total coloring of G which is proper.

Let t be a proper total coloring of G. One can verify that $t_{\mathbb{V}}$ is proper as a vertex coloring of G and $t_{\mathbb{E}}$ is proper as an edge coloring of G.

Now we state the propositions:

(151) Let us consider a subgraph H of G, and a total coloring t' of H. Suppose $t' = \langle t_{\mathbb{V}}\restriction(\text{the vertices of } H), t_{\mathbb{E}}\restriction(\text{the edges of } H)\rangle$ and t is proper. Then t' is proper. The theorem is a consequence of (15), (89), and (146).

(152) Let us consider a total coloring t^1 of G_1, and a total coloring t^2 of G_2. Suppose $G_1 \approx G_2$ and $t^1 = t^2$ and t^1 is proper. Then t^2 is proper. The theorem is a consequence of (16), (90), and (146).

(153) Let us consider a graph G_2 given by reversing directions of the edges E of G_1, a total coloring t^1 of G_1, and a total coloring t^2 of G_2. If $t^1 = t^2$, then t^1 is proper iff t^2 is proper.

(154) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, a total coloring t^1 of G_1, a total coloring t^2 of G_2, and a function h. Suppose $\operatorname{dom} h = V \setminus (\text{the vertices of } G_2)$ and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}} {+\cdot} h$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}}$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (19) and (92).

(155) Let us consider objects y, e, vertices v, w of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, a total coloring t^1 of G_1, and a total coloring t^2 of G_2. Suppose $e \notin$ the edges of G_2 and v and w are

adjacent and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}}$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}} {+\cdot} (e \dot{\longmapsto} y)$ and $y \notin \operatorname{rng} {t^2}_{\mathbb{V}} \cup \operatorname{rng} {t^2}_{\mathbb{E}}$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (20), (93), and (146).

(156) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by e between vertices v and w, a total coloring t^1 of G_1, a total coloring t^2 of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $v \neq w$ and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}} {+\cdot} (v \dot{\longmapsto} x)$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}} {+\cdot} (e \dot{\longmapsto} y)$ and $\{x, y\}$ misses $\operatorname{rng} {t^2}_{\mathbb{V}} \cup \operatorname{rng} {t^2}_{\mathbb{E}}$ and $x \neq y$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (21), (93), and (146).

(157) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by e between vertices v and w, a total coloring t^1 of G_1, a total coloring t^2 of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $v \neq w$ and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}} {+\cdot} (w \dot{\longmapsto} x)$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}} {+\cdot} (e \dot{\longmapsto} y)$ and $\{x, y\}$ misses $\operatorname{rng} {t^2}_{\mathbb{V}} \cup \operatorname{rng} {t^2}_{\mathbb{E}}$ and $x \neq y$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (156) and (153).

(158) Let us consider objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, a total coloring t^1 of G_1, a total coloring t^2 of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2 and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}} {+\cdot} (v \dot{\longmapsto} x)$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}} {+\cdot} (e \dot{\longmapsto} y)$ and $y \notin \operatorname{rng} {t^2}_{\mathbb{V}} \cup \operatorname{rng} {t^2}_{\mathbb{E}}$ and $x \neq y$ and $x \neq ({t^2}_{\mathbb{V}})(w)$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (23), (94), and (146).

(159) Let us consider a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, a total coloring t^1 of G_1, a total coloring t^2 of G_2, and objects x, y. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2 and ${t^1}_{\mathbb{V}} = {t^2}_{\mathbb{V}} {+\cdot} (w \dot{\longmapsto} x)$ and ${t^1}_{\mathbb{E}} = {t^2}_{\mathbb{E}} {+\cdot} (e \dot{\longmapsto} y)$ and $y \notin \operatorname{rng} {t^2}_{\mathbb{V}} \cup \operatorname{rng} {t^2}_{\mathbb{E}}$ and $x \neq y$ and $x \neq ({t^2}_{\mathbb{V}})(v)$ and t^2 is proper. Then t^1 is proper. The theorem is a consequence of (158) and (153).

(160) Let us consider a partial graph mapping F from G_1 to G, and a total coloring t' of G_1. Suppose F is weak subgraph embedding and $t' = \langle (t_{\mathbb{V}}) \cdot (F_{\mathbb{V}}), (t_{\mathbb{E}}) \cdot (F_{\mathbb{E}}) \rangle$ and t is proper. Then t' is proper. The theorem is a consequence of (26), (98), and (146).

Let us consider c and G. We say that G is c-total-colorable if and only if

(Def. 11) there exists a total coloring t of G such that t is proper and $\operatorname{rng} t_{\mathbb{V}} \cup \operatorname{rng} t_{\mathbb{E}} \subseteq c$.

Now we state the propositions:

(161) If $c_1 \subseteq c_2$ and G is c_1-total-colorable, then G is c_2-total-colorable.

(162) If G is c-total-colorable, then G is c-vertex-colorable and c-edge-colorable.

(163) If G is c_1-vertex-colorable and c_2-edge-colorable, then G is (c_1+c_2)-total-

colorable. The theorem is a consequence of (150) and (147).

(164) If G is edgeless and f is proper and $t = \langle f, \emptyset \rangle$, then t is proper.

(165) G is edgeless if and only if G is 1-total-colorable. The theorem is a consequence of (137) and (162).

Let c be a non zero cardinal number. One can check that there exists a graph which is c-total-colorable.

Now we state the proposition:

(166) Let us consider a subgraph H of G. If G is c-total-colorable, then H is c-total-colorable. The theorem is a consequence of (138) and (151).

Let us note that every graph is non 0-total-colorable and every graph which is edgeless is also 1-total-colorable and every graph which is 1-total-colorable is also edgeless.

Let c be a non zero cardinal number and G be a c-total-colorable graph. Note that every subgraph of G is c-total-colorable.

Let us consider c. Observe that every graph which is c-total-colorable is also loopless.

Now we state the propositions:

(167) If $G_1 \approx G_2$ and G_1 is c-total-colorable, then G_2 is c-total-colorable. The theorem is a consequence of (152).

(168) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is c-total-colorable if and only if G_2 is c-total-colorable.

Let c be a non zero cardinal number and G_1 be a c-total-colorable graph. Let us consider E. One can check that every graph given by reversing directions of the edges E of G_1 is c-total-colorable.

Now we state the proposition:

(169) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is c-total-colorable if and only if G_2 is c-total-colorable. The theorem is a consequence of (166), (139), and (154).

Let c be a non zero cardinal number and G_2 be a c-total-colorable graph. Let us consider V. Let us observe that every supergraph of G_2 extended by the vertices from V is c-total-colorable.

Now we state the propositions:

(170) Let us consider an object e, vertices v, w of G_2, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose v and w are adjacent and G_2 is c-total-colorable. Then G_1 is $(c+1)$-total-colorable. The theorem is a consequence of (141), (155), (167), and (161).

(171) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose $v \neq w$ and G_2 is c-total-colorable.

Then G_1 is $(c+2)$-total-colorable. The theorem is a consequence of (142), (156), (167), and (161).

(172) Let us consider a non edgeless graph G_2, objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. If G_2 is c-total-colorable, then G_1 is $(c+1)$-total-colorable. The theorem is a consequence of (168), (167), and (161).

(173) Let us consider a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2 and v is endvertex. If G_2 is c-total-colorable, then G_1 is c-total-colorable. The theorem is a consequence of (144) and (148).

(174) Let us consider an edgeless graph G_2, and objects v, e, w. Then every supergraph of G_2 extended by v, w and e between them is 3-total-colorable. The theorem is a consequence of (38) and (163).

(175) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose G_2 is c-total-colorable. Then G_1 is $((c+1)+\overline{\overline{V}})$-total-colorable. The theorem is a consequence of (82), (7), (96), (25), (146), (167), and (161).

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(176) If F is weak subgraph embedding and G_2 is c-total-colorable, then G_1 is c-total-colorable. The theorem is a consequence of (145) and (160).

(177) If F is isomorphism, then G_1 is c-total-colorable iff G_2 is c-total-colorable. The theorem is a consequence of (176).

Let c be a non zero cardinal number and G be a c-total-colorable graph. One can verify that every graph which is G-isomorphic is also c-total-colorable.

Let us consider G. We say that G is finitely total-colorable if and only if

(Def. 12) there exists n such that G is n-total-colorable.

Let us note that every graph which is finitely total-colorable is also loopless and every graph which is edgeless is also finitely total-colorable.

Let us consider n. One can verify that every graph which is n-total-colorable is also finitely total-colorable and there exists a graph which is finitely total-colorable and there exists a graph which is non finitely total-colorable.

Let G be a finitely total-colorable graph. One can check that every subgraph of G is finitely total-colorable.

Now we state the propositions:

(178) If $G_1 \approx G_2$ and G_1 is finitely total-colorable, then G_2 is finitely total-colorable. The theorem is a consequence of (167).

(179) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is finitely total-colorable if and only if G_2 is finitely total-colorable. The theorem is a consequence of (168).

Let G_1 be a finitely total-colorable graph. Let us consider E. Observe that every graph given by reversing directions of the edges E of G_1 is finitely total-colorable.

Let G_1 be a non finitely total-colorable graph. Note that every graph given by reversing directions of the edges E of G_1 is non finitely total-colorable.

Now we state the proposition:

(180) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1 is finitely total-colorable if and only if G_2 is finitely total-colorable. The theorem is a consequence of (169).

Let G_2 be a finitely total-colorable graph. Let us consider V. One can verify that every supergraph of G_2 extended by the vertices from V is finitely total-colorable.

Let G_2 be a non finitely total-colorable graph. Observe that every supergraph of G_2 extended by the vertices from V is non finitely total-colorable.

Now we state the propositions:

(181) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by e between vertices v and w. Suppose $v \neq w$. Then G_1 is finitely total-colorable if and only if G_2 is finitely total-colorable. The theorem is a consequence of (171).

(182) Let us consider objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then G_1 is finitely total-colorable if and only if G_2 is finitely total-colorable. The theorem is a consequence of (172) and (174).

Let G_2 be a finitely total-colorable graph and v, e, w be objects. One can check that every supergraph of G_2 extended by v, w and e between them is finitely total-colorable.

Let G_2 be a non finitely total-colorable graph. Let us observe that every supergraph of G_2 extended by v, w and e between them is non finitely total-colorable.

Now we state the proposition:

(183) Let us consider an object v, a finite set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then G_1 is finitely total-colorable if and only if G_2 is finitely total-colorable. The theorem is a consequence of (175).

Let G_2 be a finitely total-colorable graph, v be an object, and V be a finite set. Note that every supergraph of G_2 extended by vertex v and edges between

v and V of G_2 is finitely total-colorable.

Let us consider a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(184) If F is weak subgraph embedding and G_2 is finitely total-colorable, then G_1 is finitely total-colorable. The theorem is a consequence of (176).

(185) If F is isomorphism, then G_1 is finitely total-colorable iff G_2 is finitely total-colorable. The theorem is a consequence of (184).

Let G be a finitely total-colorable graph. Let us note that every graph which is G-isomorphic is also finitely total-colorable.

Let G be a graph. The functor $\chi''(G)$ yielding a cardinal number is defined by the term

(Def. 13) $\bigcap\{c$, where c is a cardinal subset of $G.\text{order}() + G.\text{size}() : G$ is c-total-colorable$\}$.

Now we state the propositions:

(186) If G is loopless, then G is $\chi''(G)$-total-colorable. The theorem is a consequence of (29), (100), and (163).

(187) G is not loopless if and only if $\chi''(G) = 0$. The theorem is a consequence of (29), (100), and (163).

Let G be a loopless graph. Let us observe that $\chi''(G)$ is non zero.

Let G be a non loopless graph. Observe that $\chi''(G)$ is zero.

Now we state the propositions:

(188) $\chi''(G) \subseteq G.\text{order}() + G.\text{size}()$. The theorem is a consequence of (29), (100), and (163).

(189) If G is c-total-colorable, then $\chi''(G) \subseteq c$. The theorem is a consequence of (188).

(190) If G is c-total-colorable and for every cardinal number d such that G is d-total-colorable holds $c \subseteq d$, then $\chi''(G) = c$. The theorem is a consequence of (189), (29), (100), and (163).

Let G be a finitely total-colorable graph. One can check that $\chi''(G)$ is natural.

Note that the functor $\chi''(G)$ yields a natural number. Now we state the propositions:

(191) $\chi(G) \subseteq \chi''(G)$. The theorem is a consequence of (186), (57), and (162).

(192) Let us consider a loopless graph G. Then $\chi'(G) \subseteq \chi''(G)$. The theorem is a consequence of (186) and (162).

(193) $\chi''(G) \subseteq \chi(G) + \chi'(G)$. The theorem is a consequence of (54), (122), (163), and (189).

(194) Let us consider a loopless graph G. Then $\bar{\Delta}(G)+1 \subseteq \chi''(G)$. The theorem is a consequence of (186), (123), and (192).

(195) G is edgeless if and only if $\chi''(G) = 1$. The theorem is a consequence of (190), (186), and (187).

(196) Let us consider a loopless, non edgeless graph G. Then $3 \subseteq \chi''(G)$. The theorem is a consequence of (195), (186), and (148).

(197) Let us consider a loopless graph G, and a subgraph H of G. Then $\chi''(H) \subseteq \chi''(G)$. The theorem is a consequence of (186) and (189).

(198) If $G_1 \approx G_2$, then $\chi''(G_1) = \chi''(G_2)$. The theorem is a consequence of (167), (186), (189), and (190).

(199) Let us consider a graph G_2 given by reversing directions of the edges E of G_1. Then $\chi''(G_1) = \chi''(G_2)$. The theorem is a consequence of (168), (186), (189), and (190).

(200) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then $\chi''(G_1) = \chi''(G_2)$. The theorem is a consequence of (169), (186), (189), and (190).

(201) Let us consider a non edgeless graph G_2, objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then $\chi''(G_1) \subseteq \chi''(G_2) + 1$. The theorem is a consequence of (186), (172), and (189).

(202) Let us consider an edgeless graph G_2, a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $w \notin$ the vertices of G_2. Then $\chi''(G_1) = 3$. The theorem is a consequence of (196), (174), and (189).

(203) Let us consider an edgeless graph G_2, objects v, e, a vertex w of G_2, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $v \notin$ the vertices of G_2. Then $\chi''(G_1) = 3$. The theorem is a consequence of (196), (174), and (189).

(204) Let us consider an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Then $\chi''(G_1) \subseteq (\chi''(G_2)+1) + \overline{\overline{V}}$. The theorem is a consequence of (186), (175), and (189).

(205) Let us consider a graph G_1, a loopless graph G_2, and a partial graph mapping F from G_1 to G_2. If F is weak subgraph embedding, then $\chi''(G_1) \subseteq \chi''(G_2)$. The theorem is a consequence of (186), (176), and (189).

(206) Let us consider a partial graph mapping F from G_1 to G_2. If F is isomorphism, then $\chi''(G_1) = \chi''(G_2)$. The theorem is a consequence of (186), (177), (189), and (190).

(207) Let us consider a G_1-isomorphic graph G_2. Then $\chi''(G_1) = \chi''(G_2)$. The theorem is a consequence of (206).

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Accepted July 23, 2022