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Transformation Tools for Real Linear Spaces

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Summary. This paper, using the Mizar system [1], [2], provides useful tools for working with real linear spaces and real normed spaces. These include the identification of a real number set with a one-dimensional real normed space, the relationships between real linear spaces and real Euclidean spaces, the transformation from a real linear space to a real vector space, and the properties of basis and dimensions of real linear spaces. We referred to [6], [10], [8], [9] in this formalization.

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1. Lipschitz Continuity of Linear Maps from Finite-Dimensional Spaces

Let n be a natural number. One can check that $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is finite dimensional. Now we state the propositions:

(1) Let us consider real linear spaces X, Y, a linear operator L from X into Y, and a finite sequence F of elements of X. Then $L(\sum F) = \sum(L \cdot F)$.
PROOF: Define $\mathcal{S}[\text{set}] \equiv$ for every finite sequence H of elements of X such that $\text{len}\, H = \$_1$ holds $L(\sum H) = \sum(L \cdot H)$. $\mathcal{S}[0]$. For every natural number n such that $\mathcal{S}[n]$ holds $\mathcal{S}[n+1]$. For every natural number n, $\mathcal{S}[n]$. □

(2) Let us consider a finite dimensional real normed space X, a real normed space Y, and a linear operator L from X into Y. If $\dim(X) \neq 0$, then L is Lipschitzian.

Proof: Set b = the ordered basis of RLSp2RVSp(X). Consider r_1, r_2 being real numbers such that $0 < r_1$ and $0 < r_2$ and for every point x of X, $\|x\| \leqslant r_1 \cdot (\text{max-norm}(X, b))(x)$ and $(\text{max-norm}(X, b))(x) \leqslant r_2 \cdot \|x\|$. Reconsider $e = b$ as a finite sequence of elements of X. Define $\mathcal{N}$(natural number) $= \|L(e_{/\$_1})\|(\in \mathbb{R})$. Consider k being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} k = \operatorname{len} b$ and for every natural number i such that $i \in \operatorname{dom} k$ holds $k(i) = \mathcal{N}(i)$. Set $k_1 = \sum k$. For every natural number i such that $i \in \operatorname{dom} k$ holds $0 \leqslant k(i)$. For every point x of X, $\|L(x)\| \leqslant r_2 \cdot (k_1 + 1) \cdot \|x\|$. □

(3) Let us consider a finite dimensional real normed space X, and a real normed space Y. Suppose $\dim(X) \neq 0$. Then LinearOperators(X, Y) = BdLinOps(X, Y). The theorem is a consequence of (2).

2. Identification of a Real Number Set with a One-Dimensional Real Normed Space

One can check that the real normed space of $\mathbb{R}$ is non empty, right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and real normed space-like. Now we state the propositions:

(4) Let us consider elements v, w of the real normed space of $\mathbb{R}$, and elements v_1, w_1 of $\mathbb{R}$. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$.

(5) Let us consider an element v of the real normed space of $\mathbb{R}$, an element v_1 of $\mathbb{R}$, and a real number a. If $v = v_1$, then $a \cdot v = a \cdot v_1$.

(6) Let us consider an element v of the real normed space of $\mathbb{R}$, and an element v_1 of $\mathbb{R}$. If $v = v_1$, then $\|v\| = |v_1|$.

3. Identification of Real Euclidean Space and Real Normed Space

Now we state the propositions:

(7) There exists a linear operator f from the real normed space of $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that

(i) f is isomorphism, and

(ii) for every element x of the real normed space of $\mathbb{R}$, $f(x) = \langle x \rangle$.

Proof: Define $\mathcal{H}$(real number) $= \langle \$_1 \rangle (\in \mathcal{R}^1)$. Consider f being a function from $\mathbb{R}$ into $\mathcal{R}^1$ such that for every element x of $\mathbb{R}$, $f(x) = \mathcal{H}(x)$. For every element x of the real normed space of $\mathbb{R}$, $f(x) = \langle x \rangle$. For every elements v,

w of the real normed space of $\mathbb{R}$, $f(v+w) = f(v)+f(w)$. For every vector x of the real normed space of $\mathbb{R}$ and for every real number r, $f(r\cdot x) = r\cdot f(x)$. For every point x of the real normed space of $\mathbb{R}$, $\|x\| = \|f(x)\|$ by [3, (1)], [5, (2)]. □

(8) (i) the real normed space of $\mathbb{R}$ is finite dimensional, and

(ii) dim(the real normed space of $\mathbb{R}$) $= 1$.

The theorem is a consequence of (7).

(9) Let us consider a real linear space sequence X, elements v, w of $\prod \overline{X}$, and an element i of dom $\overline{X}$. Then

(i) $(\prod^\circ \langle +_{X_i} \rangle_i)(v, w)(i) =$ (the addition of $X(i))(v(i), w(i))$, and

(ii) for every vectors v_2, w_2 of $X(i)$ such that $v_2 = v(i)$ and $w_2 = w(i)$ holds $(\prod^\circ \langle +_{X_i} \rangle_i)(v, w)(i) = v_2 + w_2$.

(10) Let us consider a real linear space sequence X, an element r of $\mathbb{R}$, an element v of $\prod \overline{X}$, and an element i of dom $\overline{X}$. Then

(i) $(\prod^\circ \mathrm{multop}\, X)(r, v)(i) =$ (the external multiplication of $X(i))(r, v(i))$, and

(ii) for every vector v_2 of $X(i)$ such that $v_2 = v(i)$ holds $(\prod^\circ \mathrm{multop}\, X)(r, v)(i) = r \cdot v_2$.

Let us consider a natural number n and a real norm space sequence X. Now we state the propositions:

(11) If $X = n \mapsto$ (the real normed space of $\mathbb{R}$), then $\prod X = \langle \mathcal{E}^n, \|\cdot\| \rangle$.

PROOF: Set $P_1 = \prod X$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $\overline{X}(i) = \mathbb{R}$. For every object x, $x \in \prod \overline{X}$ iff $x \in \mathcal{R}^n$. For every element j of dom $\overline{X}$, $\langle \underbrace{0, \ldots, 0}_{n} \rangle(j) = 0_{X(j)}$. For every elements a, b of $\mathcal{R}^n$, (the addition of $P_1)(a, b) = a + b$. For every real number r and for every element a of $\mathcal{R}^n$, (the external multiplication of $P_1)(r, a) = r \cdot a$. For every element a of $\mathcal{R}^n$, (the norm of $P_1)(a) = |a|$ by [4, (7)]. □

(12) Suppose $X = n \mapsto$ (the real normed space of $\mathbb{R}$). Then

(i) $\prod X$ is finite dimensional, and

(ii) $\dim(\prod X) = n$.

The theorem is a consequence of (11).

4. Transformation to Real Vector Space

Let X be a real linear space and Y be a subspace of X. One can verify that the functor RLSp2RVSp(Y) yields a subspace of RLSp2RVSp(X). Now we state the proposition:

(13) Let us consider a real linear space X, and a subspace Y of X. Then RLSp2RVSp(Y) is a subspace of RLSp2RVSp(X).

Let us consider a real linear space X and subspaces Y_1, Y_2 of X. Now we state the propositions:

(14) $\mathrm{RLSp2RVSp}(Y_1 + Y_2) = \mathrm{RLSp2RVSp}(Y_1) + \mathrm{RLSp2RVSp}(Y_2)$.

(15) $\mathrm{RLSp2RVSp}(Y_1 \cap Y_2) = \mathrm{RLSp2RVSp}(Y_1) \cap \mathrm{RLSp2RVSp}(Y_2)$.

(16) Let us consider a real linear space X.
Then $\mathrm{RLSp2RVSp}(\mathbf{0}_X) = \mathbf{0}_{\mathrm{RLSp2RVSp}(X)}$.

5. Basis and Dimension Properties of Real Linear Spaces

Now we state the propositions:

(17) Let us consider a real linear space X, and subspaces Y_1, Y_2 of X. Suppose $Y_1 \cap Y_2 = \mathbf{0}_X$. Let us consider a linearly independent subset B_1 of Y_1, and a linearly independent subset B_2 of Y_2. Then $B_1 \cup B_2$ is a linearly independent subset of $Y_1 + Y_2$. The theorem is a consequence of (15), (16), and (14).

(18) Let us consider a real linear space X, and subspaces Y_1, Y_2 of X. Suppose $Y_1 \cap Y_2 = \mathbf{0}_X$. Let us consider a basis B_1 of Y_1, and a basis B_2 of Y_2. Then $B_1 \cup B_2$ is a basis of $Y_1 + Y_2$. The theorem is a consequence of (15), (16), and (14).

(19) Let us consider real linear spaces X, Y, a subspace X_1 of X, and a subspace Y_1 of Y. Then $X_1 \times Y_1$ is a subspace of $X \times Y$.
Proof: Set $V = X \times Y$. Set $X_2 = X_1 \times Y_1$. Set $f =$ the addition of X_2. Set $g =$ (the addition of V) $\restriction$ (the carrier of X_2). For every object z such that $z \in \operatorname{dom} f$ holds $f(z) = g(z)$. Set $f =$ the external multiplication of X_2. Set $g =$ (the external multiplication of V)$\restriction$($\mathbb{R} \times$ (the carrier of X_2)). For every object z such that $z \in \operatorname{dom} f$ holds $f(z) = g(z)$. □

(20) Let us consider real linear spaces X, Y, and subspaces X_1, Y_1 of $X \times Y$. Suppose $X_1 = X \times \mathbf{0}_Y$ and $Y_1 = \mathbf{0}_X \times Y$. Then

(i) $X_1 + Y_1 = X \times Y$, and

(ii) $X_1 \cap Y_1 = \mathbf{0}_{X \times Y}$.

PROOF: For every object x, $x \in$ the carrier of $X_1 + Y_1$ iff $x \in$ the carrier of $X \times Y$. For every object x, $x \in$ (the carrier of $X \times \mathbf{0}_Y$) $\cap$ (the carrier of $\mathbf{0}_X \times Y$) iff $x \in \{\langle 0_X, 0_Y \rangle\}$ by [7, (9)]. □

Let us consider real linear spaces X, Y. Now we state the propositions:

(21) There exists a linear operator f from X into $X \times \mathbf{0}_Y$ such that

(i) f is bijective, and

(ii) for every element x of X, $f(x) = \langle x, 0_Y \rangle$.

PROOF: Set $A =$ the carrier of X. Set $B =$ the carrier of $X \times \mathbf{0}_Y$. Define $\mathcal{H}(\text{element of } A) = \langle \$_1, 0_Y \rangle (\in B)$. Consider f being a function from A into B such that for every element x of A, $f(x) = \mathcal{H}(x)$. For every element x of X, $f(x) = \langle x, 0_Y \rangle$. For every elements x_1, x_2 of X, $f(x_1 + x_2) = f(x_1) + f(x_2)$. For every vector x of X and for every real number r, $f(r \cdot x) = r \cdot f(x)$. □

(22) There exists a linear operator f from Y into $\mathbf{0}_X \times Y$ such that

(i) f is bijective, and

(ii) for every element y of Y, $f(y) = \langle 0_X, y \rangle$.

PROOF: Set $A =$ the carrier of Y. Set $B =$ the carrier of $\mathbf{0}_X \times Y$. Define $\mathcal{H}(\text{element of } A) = \langle 0_X, \$_1 \rangle (\in B)$. Consider f being a function from A into B such that for every element y of A, $f(y) = \mathcal{H}(y)$. For every element y of Y, $f(y) = \langle 0_X, y \rangle$. For every elements y_1, y_2 of Y, $f(y_1 + y_2) = f(y_1) + f(y_2)$. For every vector y of Y and for every real number r, $f(r \cdot y) = r \cdot f(y)$. □

(23) Let us consider real linear spaces X, Y, a basis B_6 of X, and a basis B_7 of Y. Then $B_6 \times \{0_Y\} \cup \{0_X\} \times B_7$ is a basis of $X \times Y$.

PROOF: Reconsider $B_4 = B_6 \times \{0_Y\}$ as a subset of the carrier of $X \times Y$. Reconsider $B_5 = \{0_X\} \times B_7$ as a subset of the carrier of $X \times Y$. Consider T_1 being a linear operator from X into $X \times \mathbf{0}_Y$ such that T_1 is bijective and for every element x of X, $T_1(x) = \langle x, 0_Y \rangle$. For every object y, $y \in T_1{}^\circ B_6$ iff $y \in B_4$.

Consider T_2 being a linear operator from Y into $\mathbf{0}_X \times Y$ such that T_2 is bijective and for every element y of Y, $T_2(y) = \langle 0_X, y \rangle$. For every object y, $y \in T_2{}^\circ B_7$ iff $y \in B_5$. Reconsider $W_1 = X \times \mathbf{0}_Y$ as a subspace of $X \times Y$. Reconsider $W_2 = \mathbf{0}_X \times Y$ as a subspace of $X \times Y$. $W_1 + W_2 = X \times Y$ and $W_1 \cap W_2 = \mathbf{0}_{X \times Y}$. □

(24) Let us consider finite dimensional real linear spaces X, Y. Then

(i) $X \times Y$ is finite dimensional, and

(ii) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

The theorem is a consequence of (23).

(25) Let us consider a finite dimensional real linear space X. Then

(i) $\prod\langle X\rangle$ is finite dimensional, and

(ii) $\dim(\prod\langle X\rangle) = \dim(X)$.

(26) Let us consider a real linear space sequence X, and a finite sequence d of elements of $\mathbb{N}$. Suppose $\operatorname{len} d = \operatorname{len} X$ and for every element i of $\operatorname{dom} X$, $X(i)$ is finite dimensional and $d(i) = \dim(X(i))$. Then

(i) $\prod X$ is finite dimensional, and

(ii) $\dim(\prod X) = \sum d$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every real linear space sequence X for every finite sequence d of elements of $\mathbb{N}$ such that $\operatorname{len} X = \$_1$ and $\operatorname{len} d = \operatorname{len} X$ and for every element i of $\operatorname{dom} X$, $X(i)$ is finite dimensional and $d(i) = \dim(X(i))$ holds $\prod X$ is finite dimensional and $\dim(\prod X) = \sum d$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

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