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Characteristic Subgroups

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Summary. We formalize in Mizar [1], [2] the notion of characteristic subgroups using the definition found in Dummit and Foote [3], as subgroups invariant under automorphisms from its parent group. Along the way, we formalize notions of Automorphism and results concerning centralizers. Much of what we formalize may be found sprinkled throughout the literature, in particular Gorenstein [4] and Isaacs [5]. We show all our favorite subgroups turn out to be characteristic: the center, the derived subgroup, the commutator subgroup generated by characteristic subgroups, and the intersection of all subgroups satisfying a generic group property.

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1. Preparatory Work

From now on X denotes a set.

Let us consider natural numbers a, b, c. Now we state the propositions:

(1) If $c \neq 0$ and $c \cdot a \mid c \cdot b$, then $a \mid b$.

(2) If $b \neq 0$ and $b \mid c$ and $a \cdot b$ and c are relatively prime, then $b = 1$.

(3) Let us consider groups G_1, G_2, a subgroup H of G_1, a homomorphism f from G_1 to G_2, and an element h of G_1. If $h \in H$, then $(f \restriction H)(h) = f(h)$.

(4) Let us consider non empty sets X, Y, and a function f from X into Y. If f is bijective, then for every element y of Y, $f((f^{-1})(y)) = y$.

(5) Let us consider non empty sets X, Y, a non empty subset A of X, and an element x of X. Suppose $x \notin A$. Let us consider a function f from X into Y. If f is one-to-one, then $f(x) \notin f^{\circ} A$.

2. Nontrivial Groups and Subgroups

Note that there exists a group which is strict and non trivial.

Let G be a group. Observe that there exists a subgroup of G which is trivial.

Let H be a subgroup of G. One can check that there exists a subgroup of H which is trivial.

Let G be a non trivial group. Observe that there exists a subgroup of G which is non trivial and there exists a subgroup of G which is strict and non trivial. Now we state the proposition:

(6) Let us consider a group G. Then G is trivial if and only if the multiplicative magma of $G = \{\mathbf{1}\}_G$.

Proof: If G is trivial, then the multiplicative magma of $G = \{\mathbf{1}\}_G$. □

Note that there exists a finite group which is non trivial.

Now we state the propositions:

(7) Let us consider a group G, and a subgroup H of G. Suppose H is trivial. Then the multiplicative magma of $H = \{\mathbf{1}\}_G$. The theorem is a consequence of (6).

(8) Let us consider a group G, a trivial subgroup H of G, and a trivial subgroup K of G. Then the multiplicative magma of H = the multiplicative magma of K. The theorem is a consequence of (7).

(9) Let us consider a group G, a trivial subgroup K of G, and a subgroup H of G. If H is a subgroup of K, then H is a trivial subgroup of G.

Proof: The carrier of $H = \{\mathbf{1}_G\}$. □

3. Proper Subgroups

Let G be a group and I_1 be a subgroup of G. We say that I_1 is proper if and only if

(Def. 1) the multiplicative magma of $I_1 \neq$ the multiplicative magma of G.

In the sequel G denotes a group and H denotes a subgroup of G.

Now we state the proposition:

(10) H is proper if and only if the carrier of $H \neq$ the carrier of G.

In the sequel h, x, y denote objects. Now we state the proposition:

(11) H is proper if and only if (the carrier of G) $\setminus$ (the carrier of H) is a non empty set. The theorem is a consequence of (10).

Let G be a non trivial group. Let us note that there exists a subgroup of G which is strict and proper and every subgroup of G which is maximal is also proper. Now we state the proposition:

(12) Let us consider a non trivial group G, a proper subgroup H of G, and a subgroup K of G. Suppose H is a subgroup of K and the multiplicative magma of $H \neq$ the multiplicative magma of K. Then K is a non trivial subgroup of G. The theorem is a consequence of (9) and (8).

4. Automorphisms

Let us consider G. An endomorphism of G is a homomorphism from G to G. From now on f denotes an endomorphism of G.

Let us consider G. One can check that there exists an endomorphism of G which is bijective.

An automorphism of G is a bijective endomorphism of G. In the sequel φ denotes an automorphism of G. Now we state the propositions:

(13) $\mathrm{Im}(f \restriction \{\mathbf{1}\}_G) = \{\mathbf{1}\}_G$.

(14) $\mathrm{Im}(\varphi \restriction \{\mathbf{1}\}_G)$ is a subgroup of $\{\mathbf{1}\}_G$. The theorem is a consequence of (13).

(15) Let us consider groups G_1, G_2, a homomorphism f from G_1 to G_2, and a subgroup H of G_1. Then $\mathrm{Ker}(f \restriction H)$ is a subgroup of $\mathrm{Ker}\, f$.

Proof: For every element g of G_1 such that $g \in \mathrm{Ker}(f \restriction H)$ holds $g \in \mathrm{Ker}\, f$. □

(16) Suppose for every automorphism f of G, $\mathrm{Im}(f \restriction H)$ is a subgroup of H. Then there exists an automorphism ψ of G such that

(i) $\psi = \varphi^{-1}$, and

(ii) $\mathrm{Im}(\varphi \restriction \mathrm{Im}(\psi \restriction H))$ is a subgroup of $\mathrm{Im}(\varphi \restriction H)$.

(17) There exists an automorphism ψ of G such that

(i) $\psi = \varphi^{-1}$, and

(ii) $\mathrm{Im}(\varphi \restriction \mathrm{Im}(\psi \restriction H)) =$ the multiplicative magma of H.

Proof: Reconsider $\psi = \varphi^{-1}$ as an automorphism of G. For every element g of G, $g \in \mathrm{Im}(\varphi \restriction \mathrm{Im}(\psi \restriction H))$ iff $g \in H$. □

(18) Let us consider a strict subgroup H of G, and a subgroup K of G. Suppose $\mathrm{Im}(\varphi \restriction H)$ is a subgroup of K. Then there exists an automorphism ψ of G such that

(i) $\psi = \varphi^{-1}$, and

(ii) H is a subgroup of $\mathrm{Im}(\psi \restriction K)$.

The theorem is a consequence of (17).

(19) H and $\varphi^{\circ} H$ are isomorphic.

(20) Let us consider a finite group G, and strict subgroups H_1, H_2 of G. Suppose H_1 and H_2 are isomorphic. Then $|\bullet : H_1|_{\mathbb{N}} = |\bullet : H_2|_{\mathbb{N}}$.

(21) Suppose G is finite. Let us consider a prime natural number p, and a strict subgroup P of G. Suppose P is a Sylow p-subgroup. Then $\mathrm{Im}(\varphi\restriction P)$ is a Sylow p-subgroup. The theorem is a consequence of (19) and (20).

(22) Let us consider an automorphism f of G. Suppose $\mathrm{Im}(f\restriction H) =$ the multiplicative magma of H. Then $f\restriction H$ is an automorphism of H.
PROOF: Set $U_H =$ the carrier of H. Reconsider $f_3 = f\restriction H$ as a function from U_H into U_H. f_3 is bijective. For every elements x, y of H, $f_3(x \cdot y) = f_3(x) \cdot f_3(y)$. □

(23) Let us consider a non trivial group G, a subgroup H of G, and an automorphism φ of G. Suppose H is a proper subgroup of G. Then $\mathrm{Im}(\varphi\restriction H)$ is a proper subgroup of G.
PROOF: Set $U_H =$ the carrier of H. Set $U_G =$ the carrier of G. $U_G \setminus U_H$ is not empty. Consider x such that $x \in U_G \setminus U_H$. $\varphi(x) \notin \varphi^\circ H$ by (5), [8, (8)]. $\varphi(x)$ is an element of G. □

(24) Let us consider a non trivial group G, a strict subgroup H of G, and an automorphism φ of G. If H is maximal, then $\mathrm{Im}(\varphi\restriction H)$ is maximal.
PROOF: $\mathrm{Im}(\varphi\restriction H)$ is a proper subgroup of G. For every strict subgroup K of G such that $\mathrm{Im}(\varphi\restriction H) \neq K$ and $\mathrm{Im}(\varphi\restriction H)$ is a subgroup of K holds $K =$ the multiplicative magma of G. □

5. Inner Automorphisms

Let us consider G. Let a be an element of G and f be a function. We say that a is inner w.r.t. f if and only if

(Def. 2) for every element x of G, $f(x) = x^a$.

Let I_1 be an automorphism of G. We say that I_1 is inner if and only if

(Def. 3) there exists an element a of G such that a is inner w.r.t. I_1.

Let G be a group and f be an automorphism of G. We introduce the notation f is outer as an antonym for f is inner.

Let us consider G. Let us observe that there exists an automorphism of G which is inner.

Let us consider a strict group G and an object f. Now we state the propositions:

(25) $f \in \mathrm{Aut}(G)$ if and only if f is an automorphism of G.

(26) $f \in \mathrm{InnAut}(G)$ if and only if f is an inner automorphism of G.

(27) Let us consider an element a of G, and an inner automorphism f of G. If a is inner w.r.t. f, then $\mathrm{Im}(f\restriction H) = H^a$.
PROOF: For every element h of G such that $h \in H$ holds $(f\restriction H)(h) = h^a$. For every element y of G such that $y \in \mathrm{Im}(f\restriction H)$ holds $y \in H^a$. For every element y of G such that $y \in H^a$ holds $y \in \mathrm{Im}(f\restriction H)$. □

Let us consider an element a of G and an endomorphism f of G. Now we state the propositions:

(28) If a is inner w.r.t. f, then $\mathrm{Ker}\, f = \{\mathbf{1}\}_G$.
PROOF: For every element x of G such that $x \in \mathrm{Ker}\, f$ holds $x \in \{\mathbf{1}\}_G$. □

(29) If a is inner w.r.t. f, then f is an automorphism of G.
PROOF: $\mathrm{Ker}\, f = \{\mathbf{1}\}_G$. There exists an endomorphism f_4 of G such that $f \cdot f_4 = \mathrm{id}_\alpha$, where α is the carrier of G. □

(30) If a is inner w.r.t. f, then f is an inner automorphism of G.

(31) Let us consider an element a of G. Then there exists an inner automorphism f of G such that a is inner w.r.t. f.
PROOF: Define $\mathcal{F}(\text{element of } G) = \$_1{}^a$. Consider f being a function from the carrier of G into the carrier of G such that for every element g of G, $f(g) = \mathcal{F}(g)$. For every elements x_1, x_2 of G, $f(x_1 \cdot x_2) = f(x_1) \cdot f(x_2)$. a is inner w.r.t. f and f is an inner automorphism of G. □

(32) Let us consider a strict subgroup H of G. Then H is normal if and only if for every inner automorphism f of G, $\mathrm{Im}(f\restriction H) = H$. The theorem is a consequence of (27) and (31).

6. Characteristic Subgroups

Let us consider G. Let I_1 be a subgroup of G. We say that I_1 is characteristic if and only if

(Def. 4) for every automorphism f of G, $\mathrm{Im}(f\restriction I_1) =$ the multiplicative magma of I_1.

Note that $\{\mathbf{1}\}_G$ is characteristic and there exists a subgroup of G which is characteristic.

From now on K denotes a characteristic subgroup of G.

Let G be a group. Let us observe that there exists a subgroup of G which is strict and characteristic. Now we state the proposition:

(33) K is a normal subgroup of G. The theorem is a consequence of (31) and (27).

Let G be a group. One can verify that every subgroup of G which is characteristic is also normal. Now we state the propositions:

(34) Let us consider groups G_1, G_2, a subgroup H_1 of G_1, a subgroup K of H_1, a subgroup H_2 of G_2, a homomorphism f from G_1 to G_2, and a homomorphism g from H_1 to H_2. Suppose for every element k of G_1 such that $k \in K$ holds $f(k) = g(k)$. Then $\mathrm{Im}(f \restriction K) = \mathrm{Im}(g \restriction K)$.
PROOF: For every object y, $y \in$ the carrier of $\mathrm{Im}(f \restriction K)$ iff $y \in$ the carrier of $\mathrm{Im}(g \restriction K)$. □

(35) Let us consider a strict subgroup H of G. Suppose for every strict subgroup K of G such that $\overline{\overline{K}} = \overline{\overline{H}}$ holds $H = K$. Then H is characteristic.
PROOF: H is characteristic. □

(36) Let us consider a strict, normal subgroup N of G. Then every characteristic subgroup of N is a normal subgroup of G.
PROOF: For every element a of G, $K^a =$ the multiplicative magma of K. □

(37) Let us consider a characteristic subgroup N of G. Then every characteristic subgroup of N is a characteristic subgroup of G.
PROOF: For every automorphism g of G, $\mathrm{Im}(g \restriction K) =$ the multiplicative magma of K. □

(38) Let us consider a group G, and a strict subgroup H of G. Then H is a characteristic subgroup of G if and only if for every automorphism φ of G, $\mathrm{Im}(\varphi \restriction H)$ is a subgroup of H.
PROOF: If H is a characteristic subgroup of G, then for every automorphism φ of G, $\mathrm{Im}(\varphi \restriction H)$ is a subgroup of H. If for every automorphism φ of G, $\mathrm{Im}(\varphi \restriction H)$ is a subgroup of H, then H is a characteristic subgroup of G. □

(39) $\mathrm{Z}(G)$ is a characteristic subgroup of G.
PROOF: Set $Z = \mathrm{Z}(G)$. For every elements y, z of G such that $z \in Z$ holds $\varphi(z) \cdot y = y \cdot \varphi(z)$. For every element z of G such that $z \in Z$ holds $(\varphi \restriction Z)(z) \in Z$. $\mathrm{Im}(\varphi \restriction Z)$ is a subgroup of Z. □

The scheme *CharMeet* deals with a group $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every automorphism φ of $\mathcal{G}$, $\varphi^\circ(\bigcap\{A$, where A is a subset of $\mathcal{G}$: there exists a strict subgroup K of $\mathcal{G}$ such that $A =$ the carrier of K and $\mathcal{P}[K]\}) = \bigcap\{A$, where A is a subset of $\mathcal{G}$: there exists a strict subgroup K of $\mathcal{G}$ such that $A =$ the carrier of K and $\mathcal{P}[K]\}$

provided

- for every automorphism φ of $\mathcal{G}$ and for every strict subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$ holds $\mathcal{P}[\mathrm{Im}(\varphi \restriction H)]$ and
- there exists a strict subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$.

The scheme *MeetIsChar* deals with a group $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) There exists a strict subgroup K of $\mathcal{G}$ such that the carrier of $K = \bigcap\{A$, where A is a subset of $\mathcal{G}$: there exists a strict subgroup H of $\mathcal{G}$ such that $A =$ the carrier of H and $\mathcal{P}[H]\}$ and K is characteristic

provided

- for every automorphism φ of $\mathcal{G}$ and for every strict subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$ holds $\mathcal{P}[\mathrm{Im}(\varphi\restriction H)]$ and
- there exists a strict subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$.

Now we state the propositions:

(40) Let us consider a non trivial group G. Suppose there exists a strict subgroup H of G such that H is maximal. Then $\Phi(G)$ is a characteristic subgroup of G.
Proof: Define $\mathcal{P}[\text{subgroup of } G] \equiv \$_1$ is maximal. For every automorphism φ of G and for every strict subgroup H of G such that $\mathcal{P}[H]$ holds $\mathcal{P}[\mathrm{Im}(\varphi\restriction H)]$. Consider K being a strict subgroup of G such that the carrier of $K = \bigcap\{A$, where A is a subset of G : there exists a strict subgroup H of G such that $A =$ the carrier of H and $\mathcal{P}[H]\}$ and K is characteristic. □

(41) Let us consider an automorphism φ of G. Then φ°(the commutators of G) = the commutators of G.
Proof: For every object g such that $g \in$ the commutators of G holds $g \in \varphi^\circ$(the commutators of G). For every object h such that $h \in \varphi^\circ$(the commutators of G) holds $h \in$ the commutators of G. □

(42) Let us consider a group G, an automorphism φ of G, and a subgroup H of G. Suppose for every element h of H, $\varphi(h) \in H$. Then $\mathrm{Im}(\varphi\restriction H)$ is a subgroup of H.
Proof: For every object y such that $y \in \mathrm{rng}(\varphi\restriction H)$ holds $y \in$ the carrier of H. □

(43) Let us consider a group G, and a non empty subset A of G. Suppose for every automorphism φ of G, $\varphi^\circ A = A$. Then $\mathrm{gr}(A)$ is characteristic.
Proof: For every automorphism φ of G and for every element a of A, $\varphi(a) \in A$. Set $H = \mathrm{gr}(A)$. For every automorphism φ of G, $\mathrm{Im}(\varphi\restriction H)$ is a subgroup of H by [7, (28)], [6, (125)]. □

(44) G^{c} is characteristic. The theorem is a consequence of (41) and (43).

Let us consider groups G_1, G_2, a subgroup H of G_1, an element a of G_1, and a homomorphism f from G_1 to G_2. Now we state the propositions:

(45) $f^\circ(a \cdot H) = f(a) \cdot (f^\circ H)$.
PROOF: For every object y such that $y \in f^\circ(a \cdot H)$ holds $y \in f(a) \cdot (f^\circ H)$. For every object y such that $y \in f(a) \cdot (f^\circ H)$ holds $y \in f^\circ(a \cdot H)$. □

(46) $f^\circ(H \cdot a) = (f^\circ H) \cdot f(a)$.
PROOF: For every object y such that $y \in f^\circ(H \cdot a)$ holds $y \in (f^\circ H) \cdot f(a)$. For every object y such that $y \in (f^\circ H) \cdot f(a)$ holds $y \in f^\circ(H \cdot a)$. □

(47) Let us consider a group G, a strict, normal subgroup N of G, and an automorphism φ of G. Then $\mathrm{Im}(\varphi \restriction N)$ is a normal subgroup of G.
PROOF: Set $H = \mathrm{Im}(\varphi \restriction N)$. For every element g of G, $g \cdot H = H \cdot g$. □

(48) Let us consider a group G, and a strict subgroup H of G. Then H is characteristic if and only if for every automorphism φ of G and for every element x of G such that $x \in H$ holds $\varphi(x) \in H$.
PROOF: If H is characteristic, then for every automorphism φ of G and for every element x of G such that $x \in H$ holds $\varphi(x) \in H$. If for every automorphism φ of G for every element x of G such that $x \in H$ holds $\varphi(x) \in H$, then H is characteristic. □

Let us consider a group G and strict, characteristic subgroups H, K of G. Now we state the propositions:

(49) $H \cap K$ is a characteristic subgroup of G.
PROOF: For every automorphism φ of G and for every element x of G such that $x \in H \cap K$ holds $\varphi(x) \in H \cap K$. □

(50) $H \sqcup K$ is a characteristic subgroup of G.
PROOF: For every automorphism φ of G and for every element g of G such that $g \in H \sqcup K$ holds $\varphi(g) \in H \sqcup K$. □

(51) Let us consider a group G, strict, characteristic subgroups H, K of G, and an automorphism φ of G. Then φ°(the commutators of H & K) = the commutators of H & K.
PROOF: For every object x such that $x \in$ the commutators of H & K holds $x \in \varphi^\circ$(the commutators of H & K). For every object y such that $y \in \varphi^\circ$(the commutators of H & K) holds $y \in$ the commutators of H & K. □

(52) Let us consider a group G, and strict, characteristic subgroups H, K of G. Then $[H, K]$ is a characteristic subgroup of G. The theorem is a consequence of (51) and (43).

7. APPENDIX 1: RESULTS CONCERNING MEETS

The scheme *MeetIsMinimal* deals with a group $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 3) There exists a strict subgroup H of $\mathcal{G}$ such that the carrier of $H = \bigcap\{A$, where A is a subset of $\mathcal{G}$: there exists a strict subgroup K of $\mathcal{G}$ suchthat $A =$ the carrier of K and $\mathcal{P}[K]\}$ and for every strict subgroup K of $\mathcal{G}$ such that $\mathcal{P}[K]$ holds H is a subgroup of K

provided

- there exists a strict subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$.

Now we state the proposition:

(53) Let us consider a group G, and subgroups H_1, H_2 of G. Suppose H_1 is a subgroup of H_2. Let us consider an element a of G. Then ${H_1}^a$ is a subgroup of ${H_2}^a$.

PROOF: For every element h of G such that $h \in {H_1}^a$ holds $h \in {H_2}^a$. □

The scheme *MeetOfNormsIsNormal* deals with a group $\mathcal{G}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 4) For every strict subgroup H of $\mathcal{G}$ such that the carrier of $H = \bigcap\{A$, where A is a subset of $\mathcal{G}$: there exists a strict subgroup N of $\mathcal{G}$ such that $A =$ the carrier of N and N is normal and $\mathcal{P}[N]\}$ holds H is a strict, normal subgroup of $\mathcal{G}$

provided

- there exists a strict, normal subgroup H of $\mathcal{G}$ such that $\mathcal{P}[H]$.

Now we state the proposition:

(54) Let us consider a group G, and a finite set X. Suppose $X \neq \emptyset$ and for every element A of X, there exists a strict, normal subgroup N of G such that $A =$ the carrier of N. Then there exists a strict, normal subgroup N of G such that the carrier of $N = \bigcap X$.

PROOF: Define $\mathcal{P}[\text{group}] \equiv \$_1$ is a normal subgroup of G and the carrier of $\$_1 \in X$. Set $F_1 = \{A$, where A is a subset of G : there exists a strict subgroup N of G such that $A =$ the carrier of N and $\mathcal{P}[N]\}$. Set $F_2 = \{A$, where A is a subset of G : there exists a strict subgroup N of G such that $A =$ the carrier of N and N is normal and $\mathcal{P}[N]\}$.

There exists a strict subgroup H of G such that $\mathcal{P}[H]$. Consider N being a strict subgroup of G such that the carrier of $N = \bigcap F_1$. For every object A, $A \in F_1$ iff $A \in F_2$. For every strict subgroup H of G such that the carrier of $H = \bigcap F_2$ holds H is a strict, normal subgroup of G. For every object A, $A \in F_1$ iff $A \in X$. □

8. Appendix 2: Centralizer of Characteristic Subgroups is Characteristic

Let G be a group and A be a subset of G. The functor Centralizer(A) yielding a strict subgroup of G is defined by

(Def. 5) the carrier of $it = \{b$, where b is an element of G : for every element a of G such that $a \in A$ holds $a \cdot b = b \cdot a\}$.

Now we state the propositions:

(55) Let us consider a group G, a subset A of G, and an element g of G. Then for every element a of G such that $a \in A$ holds $g \cdot a = a \cdot g$ if and only if g is an element of Centralizer(A).

(56) Let us consider a group G, and subsets A, B of G. Suppose $A \subseteq B$. Then Centralizer(B) is a subgroup of Centralizer(A). The theorem is a consequence of (55).

Let G be a group and H be a subgroup of G. The functor Centralizer(H) yielding a strict subgroup of G is defined by

(Def. 6) $it =$ Centralizer($\overline{H}$).

Now we state the propositions:

(57) Let us consider a group G, and a subgroup H of G. Then the carrier of Centralizer(H) $= \{b$, where b is an element of G : for every element a of G such that $a \in H$ holds $b \cdot a = a \cdot b\}$.

(58) Let us consider a group G, a subgroup H of G, and an element g of G. Then for every element a of G such that $a \in H$ holds $g \cdot a = a \cdot g$ if and only if g is an element of Centralizer(H). The theorem is a consequence of (57).

(59) Let us consider a group G. Then every subset of G is a subset of Centralizer(Centralizer(A)). The theorem is a consequence of (55) and (58).

(60) Let us consider a group G, and a strict, characteristic subgroup K of G. Then Centralizer(K) is a characteristic subgroup of G.

Proof: For every automorphism φ of G and for every element x of G such that $x \in$ Centralizer(K) holds $\varphi(x) \in$ Centralizer(K). $\square$

Let G be a group and a be an element of G. Let us observe that the functor $\{a\}$ yields a subset of G. The functor N(a) yielding a strict subgroup of G is defined by the term

(Def. 7) N($\{a\}$).

Now we state the propositions:

(61) Let us consider a group G, and elements a, x of G. Then $x \in \mathrm{N}(a)$ if and only if there exists an element h of G such that $x = h$ and $a^h = a$.

(62) Let us consider a group G, and a non empty subset A of G. Then the carrier of Centralizer(A) $= \bigcap\{B$, where B is a subset of G : there exists a strict subgroup H of G such that $B =$ the carrier of H and there exists an element a of G such that $a \in A$ and $H = \mathrm{N}(a)\}$.

Proof: Define $\mathcal{P}$[strict subgroup of G] $\equiv$ there exists an element a of G such that $a \in A$ and $\$_1 = \mathrm{N}(a)$. Set $F_1 = \{B$, where B is a subset of G : there exists a strict subgroup H of G such that $B =$ the carrier of H and $\mathcal{P}[H]\}$. $F_1 \neq \emptyset$. For every object x such that $x \in$ the carrier of Centralizer(A) holds $x \in \bigcap F_1$. For every object x such that $x \in \bigcap F_1$ holds $x \in$ the carrier of Centralizer(A). $\square$

(63) Let us consider a finite group G, and strict subgroups H_1, H_2 of G. Suppose $\overline{\overline{H_1 \cap H_2}} = \overline{\overline{H_1}}$ and $\overline{\overline{H_1 \cap H_2}} = \overline{\overline{H_2}}$. Then $H_1 = H_2$.

Proof: $H_1 \cap H_2 = H_1$. $H_1 \cap H_2 = H_2$. $\square$

(64) Let us consider finite groups G_1, G_2, a normal subgroup N_1 of G_1, and a normal subgroup N_2 of G_2. Suppose ${}^{G_1}/_{N_1}$ and ${}^{G_2}/_{N_2}$ are isomorphic. Then $\overline{\overline{N_2}} \cdot \overline{\overline{G_1}} = \overline{\overline{N_1}} \cdot \overline{\overline{G_2}}$.

(65) Let us consider a finite group G, strict, normal subgroups K, N of G, and natural numbers m, d. Suppose $m = \overline{\overline{N}}$ and $m = \overline{\overline{K}}$ and $d = \overline{\overline{K \cap N}}$. Then $d \cdot \overline{\overline{N \sqcup K}} = m \cdot m$. The theorem is a consequence of (64).

(66) Let us consider a finite group G, and a strict, normal subgroup N of G. Suppose $\overline{\overline{N}}$ and $|\bullet : N|_{\mathbb{N}}$ are relatively prime. Then N is a characteristic subgroup of G.

Proof: Consider m being a natural number such that $m = \overline{\overline{N}}$. Consider n being a natural number such that $n = |\bullet : N|_{\mathbb{N}}$. For every automorphism φ of G, $\mathrm{Im}(\varphi \restriction N) = N$. $\square$

(67) Let us consider groups G_1, G_2, G_3, a homomorphism f_1 from G_1 to G_2, a homomorphism f_2 from G_2 to G_3, and a subgroup A of G_1. Then the multiplicative magma of $f_2{}^\circ(f_1{}^\circ A)$ = the multiplicative magma of $f_2 \cdot f_1{}^\circ A$.

Proof: For every element z of G_3, $z \in f_2{}^\circ(f_1{}^\circ A)$ iff $z \in f_2 \cdot f_1{}^\circ A$. $\square$

(68) Let us consider a group G, a strict, normal subgroup N of G, and an automorphism φ of G. Suppose $\mathrm{Im}(\varphi \restriction N) = N$. Then there exists an automorphism σ of ${}^{G}/_{N}$ such that for every element x of G, $\sigma(x \cdot N) = \varphi(x) \cdot N$.

Proof: Define $\mathcal{P}[\mathrm{set}, \mathrm{set}] \equiv$ there exists an element a of G such that $\$_1 = a \cdot N$ and $\$_2 = \varphi(a) \cdot N$. For every element x of ${}^{G}/_{N}$, there exists an element y of ${}^{G}/_{N}$ such that $\mathcal{P}[x, y]$. Consider σ being a function from ${}^{G}/_{N}$ into ${}^{G}/_{N}$ such that for every element x of ${}^{G}/_{N}$, $\mathcal{P}[x, \sigma(x)]$. For every element a of G, $\sigma(a \cdot N) = \varphi(a) \cdot N$. For every elements x, y of ${}^{G}/_{N}$, $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$. σ is bijective. □

Let us consider a finite group G, a strict, characteristic subgroup H of G, and a strict subgroup K of G. Now we state the propositions:

(69) If H is a subgroup of K, then H is a normal subgroup of K.

PROOF: For every element k of K, $k \in H$ iff $k \in$ Ker((the canonical homomorphism onto cosets of $H) \restriction K$). □

(70) If H is a subgroup of K and ${}^{K}/_{(H)_K}$ is a characteristic subgroup of ${}^{G}/_{H}$, then K is a characteristic subgroup of G.

PROOF: For every automorphism φ of G and for every element k of G such that $k \in K$ holds $\varphi(k) \in K$. □

(71) Let us consider a group G, and a subgroup H of G. Then H is a subgroup of Centralizer(H) if and only if H is a commutative group.

PROOF: If H is a subgroup of Centralizer(H), then H is a commutative group. If H is a commutative group, then H is a subgroup of Centralizer(H). □

(72) Let us consider a group G. Then Centralizer(Ω_G) = Z(G).

PROOF: For every element g of G, $g \in$ Centralizer(Ω_G) iff $g \in$ Z(G). □

(73) Let us consider a group G, and a normal subgroup N of G. Then Centralizer(N) is a normal subgroup of G.

PROOF: For every elements g, n of G such that $n \in N$ holds $n^g \in N$. For every elements g, x, n of G such that $x \in$ Centralizer(N) and $n \in N$ holds $x^g \cdot n = n \cdot (x^g)$. For every elements g, z of G such that $z \in$ Centralizer(N) holds $z^g \in$ Centralizer(N). For every element g of G, (Centralizer(N))^g = Centralizer(N). □

(74) Let us consider a group G, a subgroup H of G, and elements h, n of G. If $h \in H$ and $n \in$ N(H), then $h^n \in H$.

(75) Let us consider a group G. Then every subgroup of G is a subgroup of N(H).

PROOF: For every element g of G such that $g \in H$ for every element x of G such that $x \in \overline{H}^g$ holds $x \in \overline{H}$. For every element g of G such that $g \in H$ holds $g \in$ N(H). □

(76) Let us consider a group G, and a subgroup H of G. Then Centralizer(H) is a strict, normal subgroup of N(H).

PROOF: Centralizer(H) is a normal subgroup of N(H). □

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