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Isomorphism between Spaces of Multilinear Maps and Nested Compositions over Real Normed Vector Spaces

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Summary. This paper formalizes in Mizar [1], [2], that the isometric isomorphisms between spaces formed by an $(n+1)$-dimensional multilinear map and an n-fold composition of linear maps on real normed spaces. This result is used to describe the space of nth-order derivatives of the Frechet derivative as a multilinear space. In Section 1, we discuss the spaces of 1-dimensional multilinear maps and 0-fold compositions as a preparation, and in Section 2, we extend the discussion to the spaces of $(n+1)$-dimensional multilinear map and an n-fold compositions. We referred to [4], [11], [8], [9] in this formalization.

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1. Preliminaries

Let X be a real linear space. The functor IsoCPRLSP(X) yielding a linear operator from X into $\prod\langle X\rangle$ is defined by

(Def. 1) for every point x of X, $it(x) = \langle x\rangle$.

Now we state the proposition:

(1) Let us consider a real linear space X.
Then $0_{\prod\langle X\rangle} = (\text{IsoCPRLSP}(X))(0_X)$.

Let X be a real linear space. Observe that IsoCPRLSP(X) is one-to-one and onto and there exists a linear operator from X into $\prod\langle X\rangle$ which is one-to-one and onto.

Let f be a bijective linear operator from X into $\prod\langle X\rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X\rangle$ into X. Let f be a one-to-one, onto linear operator from X into $\prod\langle X\rangle$. Let us note that f^{-1} is bijective as a linear operator from $\prod\langle X\rangle$ into X and there exists a linear operator from $\prod\langle X\rangle$ into X which is one-to-one and onto.

Now we state the propositions:

(2) Let us consider a real linear space X, and a point x of X. Then $((\text{IsoCPRLSP}(X))^{-1})(\langle x\rangle) = x$.
PROOF: Set $I = \text{IsoCPRLSP}(X)$. Set $J = I^{-1}$. For every point x of X, $J(\langle x\rangle) = x$. □

(3) Let us consider a real linear space X. Then $((\text{IsoCPRLSP}(X))^{-1})(0_{\prod\langle X\rangle}) = 0_X$. The theorem is a consequence of (1).

(4) Let us consider a real linear space G. Then

(i) for every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$, and

(ii) for every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$, and

(iii) $0_{\prod\langle G\rangle} = \langle 0_G\rangle$, and

(iv) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$, and

(v) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$.

PROOF: Consider I being a function from G into $\prod\langle G\rangle$ such that I is one-to-one and onto and for every point x of G, $I(x) = \langle x\rangle$ and for every points v, w of G, $I(v+w) = I(v)+I(w)$ and for every point v of G and for every element r of $\mathbb{R}$, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle G\rangle} = I(0_G)$. For every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$.

For every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$. □

(5) Let us consider real linear spaces X, Y, and a function f from X into Y. Then f is a linear operator from X into Y if and only if

$f \cdot ((\mathrm{IsoCPRLSP}(X))^{-1})$ is a linear operator from $\prod\langle X\rangle$ into Y.

(6) Let us consider real linear spaces X, Y, and a function f from $\prod\langle X\rangle$ into Y. Then f is a linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\mathrm{IsoCPRLSP}(X))$ is a linear operator from X into Y. The theorem is a consequence of (5).

(7) Let us consider a real linear space X, a point s of $\prod\langle X\rangle$, and an element i of $\mathrm{dom}\langle X\rangle$. Then $\mathrm{reproj}(i, s) = \mathrm{IsoCPRLSP}(X)$.
PROOF: For every element x of X, $(\mathrm{reproj}(i, s))(x) = (\mathrm{IsoCPRLSP}(X))(x)$. □

(8) Let us consider real linear spaces X, Y, and an object f. Then f is a linear operator from $\prod\langle X\rangle$ into Y if and only if f is a multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (6) and (7).

Let us consider real linear spaces X, Y. Now we state the propositions:

(9) $\mathrm{MultOpers}(\langle X\rangle, Y) = \mathrm{LinearOperators}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (8).

(10) $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) = \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (9).

(11) Let us consider a real normed space G. Then

(i) for every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$, and

(ii) for every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$, and

(iii) $0_{\prod\langle G\rangle} = \langle 0_G\rangle$, and

(iv) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$, and

(v) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$, and

(vi) for every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $\|x\| = \|x_1\|$.

PROOF: Consider I being a function from G into $\prod\langle G\rangle$ such that I is one-to-one and onto and for every point x of G, $I(x) = \langle x\rangle$ and for every points v, w of G, $I(v + w) = I(v) + I(w)$ and for every point v of G and for every element r of $\mathbb{R}$, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle G\rangle} = I(0_G)$ and for every point v of G, $\|I(v)\| = \|v\|$. For every set x, x is a point of $\prod\langle G\rangle$ iff there exists a point x_1 of G such that $x = \langle x_1\rangle$.

For every points x, y of $\prod\langle G\rangle$ and for every points x_1, y_1 of G such that $x = \langle x_1\rangle$ and $y = \langle y_1\rangle$ holds $x + y = \langle x_1 + y_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $-x = \langle -x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G and for every real number a such that $x = \langle x_1\rangle$ holds $a \cdot x = \langle a \cdot x_1\rangle$. For every point x of $\prod\langle G\rangle$ and for every point x_1 of G such that $x = \langle x_1\rangle$ holds $\|x\| = \|x_1\|$. □

Let X be a real normed space. The functor IsoCPNrSP(X) yielding a linear operator from X into $\prod\langle X\rangle$ is defined by

(Def. 2) for every point x of X, $it(x) = \langle x\rangle$.

Now we state the proposition:

(12) Let us consider a real normed space X.
Then $0_{\prod\langle X\rangle} = (\text{IsoCPNrSP}(X))(0_X)$.

Let X be a real normed space. Let us note that IsoCPNrSP(X) is one-to-one, onto, and isometric and there exists a linear operator from X into $\prod\langle X\rangle$ which is one-to-one, onto, and isometric.

Let I be a one-to-one, onto, isometric linear operator from X into $\prod\langle X\rangle$. Let us observe that the functor I^{-1} yields a linear operator from $\prod\langle X\rangle$ into X. One can check that I^{-1} is one-to-one, onto, and isometric as a linear operator from $\prod\langle X\rangle$ into X and there exists a linear operator from $\prod\langle X\rangle$ into X which is one-to-one, onto, and isometric. Let us consider real normed spaces X, Y and a function f from X into Y. Now we state the propositions:

(13) f is a linear operator from X into Y if and only if $f \cdot ((\text{IsoCPNrSP}(X))^{-1})$ is a linear operator from $\prod\langle X\rangle$ into Y.

(14) f is a Lipschitzian linear operator from X into Y if and only if $f \cdot ((\text{IsoCPNrSP}(X))^{-1})$ is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y.

Let us consider real normed spaces X, Y and a function f from $\prod\langle X\rangle$ into Y. Now we state the propositions:

(15) f is a linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\text{IsoCPNrSP}(X))$ is a linear operator from X into Y. The theorem is a consequence of (13).

(16) f is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y if and only if $f \cdot (\text{IsoCPNrSP}(X))$ is a Lipschitzian linear operator from X into Y. The theorem is a consequence of (14).

(17) Let us consider a real normed space X, a point s of $\prod\langle X\rangle$, and an element i of dom$\langle X\rangle$. Then reproj(i, s) = IsoCPNrSP(X).
PROOF: For every element x of X, $(\text{reproj}(i, s))(x) = (\text{IsoCPNrSP}(X))(x)$. □

(18) Let us consider a real normed space X, and a point x of $\prod\langle X\rangle$. Then NrProduct $x = \|x\|$. The theorem is a consequence of (11).

Let us consider real normed spaces X, Y and an object f. Now we state the propositions:

(19) f is a linear operator from $\prod\langle X\rangle$ into Y if and only if f is a multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (15) and (17).

(20) f is a Lipschitzian linear operator from $\prod\langle X\rangle$ into Y if and only if f is a Lipschitzian multilinear operator from $\langle X\rangle$ into Y. The theorem is a consequence of (16), (18), (17), and (11).

Let us consider real normed spaces X, Y. Now we state the propositions:

(21) $\mathrm{MultOpers}(\langle X\rangle, Y) = \mathrm{LinearOperators}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (19).

(22) $\mathrm{BoundedMultOpers}(\langle X\rangle, Y) = \mathrm{BdLinOps}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (20).

(23) $\mathrm{BoundedMultOpersNorm}(\langle X\rangle, Y) = \mathrm{BdLinOpsNorm}(\prod\langle X\rangle, Y)$.
Proof: Set $n_1 = \mathrm{BoundedMultOpersNorm}(\langle X\rangle, Y)$. Set $n_2 = \mathrm{BdLinOpsNorm}(\prod\langle X\rangle, Y)$. $\mathrm{BoundedMultOpers}(\langle X\rangle, Y) = \mathrm{BdLinOps}(\prod\langle X\rangle, Y)$. For every object f such that $f \in \mathrm{BoundedMultOpers}(\langle X\rangle, Y)$ holds $n_1(f) = n_2(f)$. □

(24) $\mathrm{VectorSpaceOfMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) = \mathrm{VectorSpaceOfLinearOpers}_{\mathbb{R}}(\prod\langle X\rangle, Y)$. The theorem is a consequence of (21).

(25) $\mathrm{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(\langle X\rangle, Y) =$ the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y. The theorem is a consequence of (24) and (23).

(26) Let us consider a real normed space X. If X is complete, then $\prod\langle X\rangle$ is complete.

2. Spaces of Multilinear Maps and Nested Compositions over Real Normed Vector Spaces

Now we state the propositions:

(27) Let us consider real norm space sequences X, Y, a real normed space Z, and a Lipschitzian bilinear operator f from $\prod X \times \prod Y$ into Z. Then $f \cdot ((\mathrm{IsoCPNrSP}(\prod X, \prod Y))^{-1})$ is a Lipschitzian multilinear operator from $\langle \prod X, \prod Y\rangle$ into Z.

(28) Let us consider real norm space sequences X, Y, a real normed space Z, and a point f of $\mathrm{NormSpaceOfBoundedBilinOpers}_{\mathbb{R}}(\prod X, \prod Y, Z)$. Then $f \cdot ((\mathrm{IsoCPNrSP}(\prod X, \prod Y))^{-1})$ is a point of NormSpaceOfBoundedMult-$\mathrm{Opers}_{\mathbb{R}}(\langle \prod X, \prod Y\rangle, Z)$.

(29) Let us consider real linear space sequences X, Y. Then $\overline{X \frown Y} = \overline{X} \frown \overline{Y}$.
PROOF: Reconsider $C_1 = \overline{X}$, $C_2 = \overline{Y}$ as a finite sequence. For every natural number i such that $i \in \operatorname{dom} \overline{X \frown Y}$ holds $\overline{X \frown Y}(i) = (C_1 \frown C_2)(i)$. $\square$

(30) Let us consider a real linear space X. Then

(i) $\operatorname{len} \overline{\langle X \rangle} = \operatorname{len} \langle X \rangle$, and

(ii) $\operatorname{len} \overline{\langle X \rangle} = 1$, and

(iii) $\overline{\langle X \rangle} = \langle$the carrier of $X\rangle$.

(31) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, an element i of $\operatorname{dom} X$, an element j of $\operatorname{dom}(X \frown \langle Y \rangle)$, an element x_i of $X(i)$, and a point y of Y. Suppose $i = j$ and $z = x \frown \langle y \rangle$. Then $(\operatorname{reproj}(j, z))(x_i) = (\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle$.
PROOF: Reconsider $x_j = x_i$ as an element of $(X \frown \langle Y \rangle)(j)$. For every object k such that $k \in \operatorname{dom}((\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle)$ holds $((\operatorname{reproj}(i, x))(x_i) \frown \langle y \rangle)(k) = (\operatorname{reproj}(j, z))(x_j)(k)$. $\square$

(32) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, an element j of $\operatorname{dom}(X \frown \langle Y \rangle)$, an element y of Y, and a point y_0 of Y. Suppose $z = x \frown \langle y_0 \rangle$ and $j = \operatorname{len} x + 1$. Then $(\operatorname{reproj}(j, z))(y) = x \frown \langle y \rangle$.
PROOF: Reconsider $y_1 = y$ as an element of $(X \frown \langle Y \rangle)(j)$. For every object k such that $k \in \operatorname{dom}((\operatorname{reproj}(j, z))(y_1))$ holds $(\operatorname{reproj}(j, z))(y_1)(k) = (x \frown \langle y \rangle)(k)$. $\square$

(33) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, and a point y of Y. Then $x \frown \langle y \rangle$ is a point of $\prod(X \frown \langle Y \rangle)$.
PROOF: Set $C_1 = \overline{X}$. Set $C_2 =$ the carrier of Y. The carrier of $\prod(X \frown \langle Y \rangle) = \prod(\overline{X} \frown \overline{\langle Y \rangle})$. For every object i such that $i \in \operatorname{dom}(C_1 \frown \langle C_2 \rangle)$ holds $(x \frown \langle y \rangle)(i) \in (C_1 \frown \langle C_2 \rangle)(i)$. $\square$

(34) Let us consider a real norm space sequence X, an element x of $\prod X$, a real normed space Y, an element z of $\prod(X \frown \langle Y \rangle)$, and a point y of Y. Suppose $z = x \frown \langle y \rangle$. Then $\operatorname{NrProduct} z = \|y\| \cdot (\operatorname{NrProduct} x)$.
PROOF: Consider n_4 being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} n_4 = \operatorname{dom}(X \frown \langle Y \rangle)$ and for every element i of $\operatorname{dom}(X \frown \langle Y \rangle)$, $n_4(i) = \|z(i)\|$ and $\operatorname{NrProduct} z = \prod n_4$. Set $n_3 = n_4 \restriction \operatorname{len} x$. Set $C_1 = \overline{X}$. Consider x_1 being a function such that $x = x_1$ and $\operatorname{dom} x_1 = \operatorname{dom} C_1$ and for every object i such that $i \in \operatorname{dom} C_1$ holds $x_1(i) \in C_1(i)$. For every element i of $\operatorname{dom} X$, $n_3(i) = \|x(i)\|$. $0 \leqslant \prod n_3$ by [7, (42)]. For every object i such that $i \in \operatorname{dom}(n_3 \frown \langle \|y\| \rangle)$ holds $(n_3 \frown \langle \|y\| \rangle)(i) = n_4(i)$. $\square$

(35) Let us consider real normed spaces X, Z, and a real norm space sequence Y. Then there exists a Lipschitzian linear operator I from the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$ into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$ such that

(i) I is one-to-one, onto, and isometric, and

(ii) for every point u of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$, $\|u\| = \|I(u)\|$ and for every point y of $\prod Y$ and for every point x of X, $I(u)(y \frown \langle x \rangle) = u(x)(y)$.

Proof: Set C_1 = the carrier of X. Set $C_2 = \overline{Y}$. Set C_3 = the carrier of Z. Consider J being a function from $(C_3{}^{\prod C_2})^{C_1}$ into $C_3{}^{\prod (C_2 \frown \langle C_1 \rangle)}$ such that J is bijective and for every function f from C_1 into $C_3{}^{\prod C_2}$ and for every finite sequence y and for every object x such that $y \in \prod C_2$ and $x \in C_1$ holds $J(f)(y \frown \langle x \rangle) = f(x)(y)$. Set L_1 = the carrier of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}$ (Y, Z). Set B_1 = the carrier of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$. Set L_2 = the carrier of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$. The carrier of $\prod \langle X \rangle = \prod \langle$the carrier of $X\rangle$. The carrier of $\prod (Y \frown \langle X \rangle) = \prod (\overline{Y} \frown \overline{\langle X \rangle})$. ${L_2}^{C_1} \subseteq (C_3{}^{\prod C_2})^{C_1}$. Reconsider $I = J \restriction L_1$ as a function from L_1 into $C_3{}^{\prod (C_2 \frown \langle C_1 \rangle)}$.

For every element f of L_1, for every point x of X, there exists a Lipschitzian multilinear operator g from Y into Z such that $g = f(x)$ and for every point y of $\prod Y$, $I(f)(y \frown \langle x \rangle) = g(y)$ and $I(f)$ is a Lipschitzian multilinear operator from $Y \frown \langle X \rangle$ into Z and $I(f) \in B_1$ and there exists a point I_f of NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y \frown \langle X \rangle, Z)$ such that $I_f = I(f)$ and $\|f\| = \|I_f\|$. For every elements f_1, f_2 of L_1, $I(f_1 + f_2) = I(f_1) + I(f_2)$. For every element f_1 of L_1 and for every real number a, $I(a \cdot f_1) = a \cdot I(f_1)$ by [6, (2)], (11), [5, (49)]. For every point u of the real norm space of bounded linear operators from X into NormSpaceOfBoundedMultOpers$_{\mathbb{R}}(Y, Z)$, $\|u\| = \|I(u)\|$ and for every point y of $\prod Y$ and for every point x of X, $I(u)(y \frown \langle x \rangle) = u(x)(y)$. For every object I_f such that $I_f \in B_1$ there exists an object f such that $f \in L_1$ and $I_f = I(f)$. $\square$

Let Y be a real normed space and X be a real norm space sequence. The functor NestingLB(X, Y) yielding a real normed space is defined by

(Def. 3) there exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and *it* $= f(\operatorname{len} X)$ and $f(0) = Y$ and for every natural number i such that $i < \operatorname{len} X$ there exists a real normed space f_i and there exists an element j of $\operatorname{dom} X$ such that

$f_i = f(i)$ and $i + 1 = j$ and $f(i + 1) =$ the real norm space of bounded linear operators from $X(j)$ into f_i.

Let us consider real normed spaces X, Y, Z and a Lipschitzian linear operator I from Y into Z. Now we state the propositions:

(36) Suppose I is one-to-one, onto, and isometric. Then there exists a Lipschitzian linear operator L from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from X into Z such that

(i) L is one-to-one, onto, and isometric, and

(ii) for every point f of the real norm space of bounded linear operators from X into Y, $L(f) = I \cdot f$.

PROOF: Consider J being a linear operator from Z into Y such that $J = I^{-1}$ and J is one-to-one and onto and J is isometric. Set $F =$ the carrier of the real norm space of bounded linear operators from X into Y. Set $G =$ the carrier of the real norm space of bounded linear operators from X into Z. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = I \cdot \$_1$. For every element f of F, there exists an element g of G such that $\mathcal{P}[f, g]$. Consider L being a function from F into G such that for every element f of F, $\mathcal{P}[f, L(f)]$.

For every objects f_1, f_2 such that f_1, $f_2 \in F$ and $L(f_1) = L(f_2)$ holds $f_1 = f_2$. For every object g such that $g \in G$ there exists an object f such that $f \in F$ and $g = L(f)$ by [10, (2)]. For every points f_1, f_2 of the real norm space of bounded linear operators from X into Y, $L(f_1 + f_2) = L(f_1) + L(f_2)$. For every point f of the real norm space of bounded linear operators from X into Y and for every real number a, $L(a \cdot f) = a \cdot L(f)$. For every element f of the real norm space of bounded linear operators from X into Y, $\|L(f)\| = \|f\|$ by [3, (7)]. □

(37) Suppose I is one-to-one, onto, and isometric. Then there exists a Lipschitzian linear operator L from the real norm space of bounded linear operators from Y into X into the real norm space of bounded linear operators from Z into X such that

(i) L is one-to-one, onto, and isometric, and

(ii) for every point f of the real norm space of bounded linear operators from Y into X, $L(f) = f \cdot (I^{-1})$.

PROOF: Consider J being a linear operator from Z into Y such that $J = I^{-1}$ and J is one-to-one and onto and J is isometric. Set $F =$ the carrier of the real norm space of bounded linear operators from Y into X. Set $G =$ the carrier of the real norm space of bounded linear operators from Z into X. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = \$_1 \cdot J$. For every element f

of F, there exists an element g of G such that $\mathcal{P}[f, g]$. Consider L being a function from F into G such that for every element f of F, $\mathcal{P}[f, L(f)]$.

For every objects f_1, f_2 such that f_1, $f_2 \in F$ and $L(f_1) = L(f_2)$ holds $f_1 = f_2$. For every object g such that $g \in G$ there exists an object f such that $f \in F$ and $g = L(f)$. For every points f_1, f_2 of the real norm space of bounded linear operators from Y into X, $L(f_1 + f_2) = L(f_1) + L(f_2)$. For every point f of the real norm space of bounded linear operators from Y into X and for every real number a, $L(a \cdot f) = a \cdot L(f)$. For every element f of the real norm space of bounded linear operators from Y into X, $\|L(f)\| = \|f\|$. □

(38) Let us consider real normed spaces X, Y. Then there exists a Lipschitzian linear operator I from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y such that

(i) I is one-to-one, onto, and isometric, and

(ii) for every point u of the real norm space of bounded linear operators from X into Y and for every point x of X, $I(u)(\langle x\rangle) = u(x)$, and

(iii) for every point u of the real norm space of bounded linear operators from X into Y, $\|u\| = \|I(u)\|$.

Proof: Set $J = \mathrm{IsoCPNrSP}(X)$. Consider I being a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from $\prod\langle X\rangle$ into Y such that I is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from X into Y, $I(x) = x \cdot (J^{-1})$. For every point u of the real norm space of bounded linear operators from X into Y and for every point x of X, $I(u)(\langle x\rangle) = u(x)$. □

(39) Let us consider real normed spaces X, Y, Z, W, a Lipschitzian linear operator I from X into Z, and a Lipschitzian linear operator J from Y into W. Suppose I is one-to-one, onto, and isometric and J is one-to-one, onto, and isometric.

Then there exists a Lipschitzian linear operator K from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into W such that

(i) K is one-to-one, onto, and isometric, and

(ii) for every point x of the real norm space of bounded linear operators from X into Y, $K(x) = J \cdot (x \cdot (I^{-1}))$.

Proof: Consider H being a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into Y such that H is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from X into Y, $H(x) = x \cdot (I^{-1})$. Consider L being a Lipschitzian linear operator from the real norm space of bounded linear operators from Z into Y into the real norm space of bounded linear operators from Z into W such that L is one-to-one, onto, and isometric and for every point x of the real norm space of bounded linear operators from Z into Y, $L(x) = J \cdot x$.

Reconsider $K = L \cdot H$ as a Lipschitzian linear operator from the real norm space of bounded linear operators from X into Y into the real norm space of bounded linear operators from Z into W. For every point x of the real norm space of bounded linear operators from X into Y, $\|K(x)\| = \|x\|$. □

(40) Let us consider a natural number n, real norm space sequences A, B, and real normed spaces X, Y. Suppose $\operatorname{len} A = n + 1$ and $A{\restriction}n = B$ and $X = A(n+1)$. Then $\operatorname{NestingLB}(A, Y) =$ the real norm space of bounded linear operators from X into $\operatorname{NestingLB}(B, Y)$.

PROOF: Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and NestingLB $(A, Y) = f(\operatorname{len} A)$ and $f(0) = Y$ and for every natural number j such that $j < \operatorname{len} A$ there exists a real normed space V and there exists an element k of $\operatorname{dom} A$ such that $V = f(j)$ and $j + 1 = k$ and $f(j + 1) =$ the real norm space of bounded linear operators from $A(k)$ into V.

Consider V being a real normed space, k being an element of $\operatorname{dom} A$ such that $V = f(\operatorname{len} B)$ and $\operatorname{len} B+1 = k$ and $f(\operatorname{len} B+1) =$ the real norm space of bounded linear operators from $A(k)$ into V. For every natural number j such that $j < \operatorname{len} B$ there exists a real normed space V and there exists an element k of $\operatorname{dom} B$ such that $V = f(j)$ and $j + 1 = k$ and $f(j + 1) =$ the real norm space of bounded linear operators from $B(k)$ into V. □

Let Y be a real normed space and X be a real norm space sequence. Let us observe that $\operatorname{NestingLB}(X, Y)$ is constituted functions.

The functor $\operatorname{NestMult}(X, Y)$ yielding a Lipschitzian linear operator from $\operatorname{NestingLB}(X, Y)$ into $\operatorname{NormSpaceOfBoundedMultOpers}_{\mathbb{R}}(X, Y)$ is defined by

(Def. 4) *it* is one-to-one, onto, and isometric and for every element u of NestingLB (X, Y), $\|it(u)\| = \|u\|$ and for every point u of $\operatorname{NestingLB}(X, Y)$ and for every point x of $\prod X$, there exists a finite sequence g such that $\operatorname{len} g = \operatorname{len} X$ and $g(1) = u$ and for every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} X$ there exists a real norm space sequence X_2.

There exists a point h of $\operatorname{NestingLB}(X_2, Y)$ such that $X_2 = X{\restriction}(\operatorname{len} X -' i+1)$ and $h = g(i)$ and $g(i+1) = h(x(\operatorname{len} X -' i+1))$ and there exists a real norm space sequence X_1 and there exists a point h of NestingLB(X_1, Y) such that $X_1 = \langle X(1) \rangle$ and $h = g(\operatorname{len} X)$ and $(it(u))(x) = h(x(1))$.

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