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Non-Trivial Universes and Sequences of Universes[1]

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Summary. Universe is a concept which is present from the beginning of the creation of the Mizar Mathematical Library (MML) in several forms (`Universe`, `Universe_closure`, `UNIVERSE`) [25], then later as `the_universe_of`, [33], and recently with the definition `GrothendieckUniverse` [26], [11], [11]. These definitions are useful in many articles [28, 33, 8, 35], [19, 32, 31, 15, 6], but also [34, 12, 20, 22, 21], [27, 2, 3, 23, 16, 7, 4, 5].

In this paper, using the Mizar system [9] [10], we trivially show that Grothendieck's definition of Universe as defined in [26], coincides with the original definition of Universe defined by Artin, Grothendieck, and Verdier (*Chapitre 0 Univers et Appendice "Univers" (par N. Bourbaki) de l'Exposé I. "PREFAISCEAUX"*) [1], and how the different definitions of MML concerning universes are related. We also show that the definition of Universe introduced by Mac Lane ([18]) is compatible with the MML's definition.

Although a universe may be empty, we consider the properties of non-empty universes, completing the properties proved in [25].

We introduce the notion of "trivial" and "non-trivial" Universes, depending on whether or not they contain the set ω (`NAT`), following the notion of Robert M. Solovay[2]. The following result links the universes $\mathbf{U}_0$ (`FinSETS`) and $\mathbf{U}_1$ (`SETS`):

$$\text{GrothendieckUniverse}\ \omega = \text{GrothendieckUniverse}\ \mathbf{U}_0 = \mathbf{U}_1$$

Before turning to the last section, we establish some trivial propositions allowing the construction of sets outside the considered universe.

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[2]`https://cs.nyu.edu/pipermail/fom/2008-March/012783.html`

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The last section is devoted to the construction, in Tarski-Grothendieck, of a tower of universes indexed by the ordinal numbers (See 8. Examples, Grothendieck universe, ncatlab.org [24]).

Grothendieck's universe is referenced in current works: "Assuming the existence of a sufficient supply of (Grothendieck) univers", Jacob Lurie in "Higher Topos Theory" [17], "Annexe B – Some results on Grothendieck universes", Olivia Caramello and Riccardo Zanfa in "Relative topos theory via stacks" [13], "Remark 1.1.5 (quoting Michael Shulman [30])", Emily Riehl in "Category theory in Context" [29], and more specifically "Strict Universes for Grothendieck Topoi" [14].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a set X. Then $\pi_1(X)$, $\pi_2(X) \in 2^{\bigcup \bigcup X}$.

(2) $\mathbb{R}^* =$ the set of all X where X is a finite sequence of elements of $\mathbb{R}$.

One can verify that there exists a Grothendieck which is empty and there exists a Grothendieck which is non empty.

Let X be a set. One can verify that every Grothendieck of X is non empty.

2. Original Definitions of Grothendieck's Universe

Let $\mathcal{G}$ be a set. We say that $\mathcal{G}$ satisfies axiom GU_1 if and only if

(Def. 1) for every sets x, y such that $x \in \mathcal{G}$ and $y \in x$ holds $y \in \mathcal{G}$.

We say that $\mathcal{G}$ satisfies axiom GU_2 if and only if

(Def. 2) for every sets x, y such that $x, y \in \mathcal{G}$ holds $\{x, y\} \in \mathcal{G}$.

We say that $\mathcal{G}$ satisfies axiom GU_3 if and only if

(Def. 3) for every set x such that $x \in \mathcal{G}$ holds $2^x \in \mathcal{G}$.

Let $\mathcal{G}$ be a non empty set. We say that $\mathcal{G}$ satisfies axiom GU_4 if and only if

(Def. 4) for every element I of $\mathcal{G}$ and for every $\mathcal{G}$-valued many sorted set x indexed by I, $\bigcup \operatorname{rng} x \in \mathcal{G}$.

3. Equivalences of Definitions

Now we state the propositions:

(3) Let us consider a set X. Then X satisfies axiom GU_1 if and only if X is transitive.

(4) Let us consider a non empty set X. Then X satisfies axiom GU_4 if and only if X is Family-Union-closed.

(5) Let us consider a Family-Union-closed set X, and a function f. Suppose $\mathrm{dom}\, f \in X$ and $\mathrm{rng}\, f \subseteq X$. Then $\bigcup \mathrm{rng}\, f \in X$.

One can check that every Grothendieck satisfies axiom GU_1, axiom GU_2, and axiom GU_3 and every non empty Grothendieck satisfies axiom GU_4.

Now we state the proposition:

(6) Let us consider a non empty set $\mathcal{G}$. Suppose $\mathcal{G}$ satisfies axiom GU_1, axiom GU_2, axiom GU_3, and axiom GU_4. Then $\mathcal{G}$ is a non empty Grothendieck.

Let us consider a set X. Now we state the propositions:

(7) X is a universal class if and only if X is a non empty Grothendieck.

(8) $\mathbf{T}(\{X\}^{*\in})$ is a Grothendieck of X.

(9) The universe of $\{X\}$ is a Grothendieck of X. The theorem is a consequence of (8).

(10) Universe_closure($\{X\}$) = GrothendieckUniverse(X).

4. Equivalences of Mac Lane Definition

Now we state the propositions:

(11) Let us consider a Grothendieck U. Suppose $\omega \in U$. Then

  (i) for every sets x, u such that $x \in u \in U$ holds $x \in U$, and

  (ii) for every sets u, v such that $u, v \in U$ holds $\{u, v\}$, $\langle u, v\rangle$, $u \times v \in U$, and

  (iii) for every set x such that $x \in U$ holds 2^x, $\bigcup x \in U$, and

  (iv) $\omega \in U$, and

  (v) for every sets a, b and for every function f from a into b such that $\mathrm{dom}\, f = a$ and f is onto and $a \in U$ and $b \subseteq U$ holds $b \in U$.

(12) Let us consider a set U. Suppose for every sets x, u such that $x \in u \in U$ holds $x \in U$ and for every set x such that $x \in U$ holds 2^x, $\bigcup x \in U$ and $\omega \in U$ and for every sets a, b and for every function f from a into b such that $\mathrm{dom}\, f = a$ and f is onto and $a \in U$ and $b \subseteq U$ holds $b \in U$. Then U is a Grothendieck. The theorem is a consequence of (4) and (3).

5. Properties of Universe, Following [25]

From now on X denotes a set and $\mathcal{U}$ denotes a universal class.

Now we state the proposition:

(13) Suppose X satisfies axiom GU_1 and axiom GU_3. Then

(i) for every set y and for every subset x of y such that $y \in X$ holds $x \in X$, and

(ii) for every sets x, y such that $x \subseteq y$ and $y \in X$ holds $x \in X$, and

(iii) if X is not empty, then $\emptyset \in X$.

Let $\mathcal{U}$ be a universal class. The functor $\emptyset_{\mathcal{U}}$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 5) $\emptyset$.

Now we state the propositions:

(14) $\mathcal{U}$ is a Grothendieck of $\emptyset$. The theorem is a consequence of (13).

(15) Let us consider elements u, v of $\mathcal{U}$. Then $v^u \subseteq$ the set of all f where f is a function from u into v.

Let $\mathcal{U}$ be a universal class and u be an element of $\mathcal{U}$. Note that the functor $\operatorname{succ} u$ yields an element of $\mathcal{U}$. Now we state the propositions:

(16) Let us consider a natural number n. Then $n \in \mathcal{U}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 \in \mathcal{U}$. $\mathcal{P}[0]$. For every natural number n, $\mathcal{P}[n]$. □

(17) $\omega \subseteq \mathcal{U}$.

(18) (i) $\mathbb{N} \in \mathcal{U}$, or

(ii) $\mathbb{N} \approx \mathcal{U}$.

The theorem is a consequence of (16).

Let us note that every universal class is infinite. Now we state the proposition:

(19) $\mathbf{U}_0$ is denumerable.

Observe that there exists a universal class which is denumerable.

Now we state the proposition:

(20) $\mathcal{U}$ is not denumerable if and only if $\omega \in \mathcal{U}$.

Observe that there exists a universal class which is non denumerable.

Let $\mathcal{U}$ be a universal class. We say that $\mathcal{U}$ is trivial if and only if

(Def. 6) $\omega \notin \mathcal{U}$.

Now we state the proposition:

(21) (i) $\mathbf{U}_0$ is trivial, and

(ii) $\mathbf{U}_1$ is not trivial.

The theorem is a consequence of (16), (13), (19), and (20).

One can check that there exists a universal class which is trivial and there exists a universal class which is non trivial and every non trivial universal class is non denumerable. Now we state the proposition:

(22) Let us consider an element x of $\mathcal{U}$, and objects y, z. Suppose $x = \langle y, z \rangle$. Then

(i) y is an element of $\mathcal{U}$, and

(ii) z is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class. Let us note that there exists an element of $\mathcal{U}$ which is pair. Now we state the proposition:

(23) Let us consider elements u, v of $\mathcal{U}$. Then the set of all f where f is a function from u into v is an element of $\mathcal{U}$. The theorem is a consequence of (13).

Let $\mathcal{U}$ be a universal class, I be an element of $\mathcal{U}$, and x be a $\mathcal{U}$-valued many sorted set indexed by I. Let us observe that the functor $\prod x$ yields an element of $\mathcal{U}$. Let x, y be elements of $\mathcal{U}$. The functor $x \uplus y$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 7) $[x \longmapsto \emptyset_{\mathcal{U}}, y \longmapsto \{\emptyset_{\mathcal{U}}\}]$.

Now we state the propositions:

(24) Let us consider elements x, y of $\mathcal{U}$. Then $x \uplus y$ is a subset of $\{x, y\} \times \{\emptyset, \{\emptyset\}\}$.

(25) Let us consider an element u of $\mathcal{U}$. Then $u \uplus u = \{\langle u, \{\emptyset\}\rangle\}$.

Let $\mathcal{U}$ be a universal class, I be an element of $\mathcal{U}$, and x be a $\mathcal{U}$-valued many sorted set indexed by I. Note that the functor $\operatorname{dom} x$ yields an element of $\mathcal{U}$. Note that the functor $\bigcup x$ yields an element of $\mathcal{U}$. Let us note that the functor $\operatorname{disjoint} x$ yields a $\mathcal{U}$-valued many sorted set indexed by I. The functor $\biguplus x$ yielding an element of $\mathcal{U}$ is defined by the term

(Def. 8) $\bigcup \operatorname{disjoint} x$.

Let us consider an element I of $\mathcal{U}$ and a $\mathcal{U}$-valued many sorted set x indexed by I. Now we state the propositions:

(26) $\bigcup \operatorname{coprod}(x)$ is an element of $\mathcal{U}$.

(27) $\biguplus x$ is a subset of $\bigcup \operatorname{rng} x \times I$.

(28) If X satisfies axiom GU_2, then for every set x such that $x \in X$ holds $\{x\} \in X$.

Let us consider an element u of $\mathcal{U}$. Now we state the propositions:

(29) $\overline{\overline{u}} \in \mathcal{U}$.

(30) (i) $u \not\approx \mathcal{U}$, and

(ii) $\overline{\overline{u}} \in \overline{\overline{\mathcal{U}}}$.

(31) Let us consider elements u, v of $\mathcal{U}$. Then $\{\langle u, \emptyset\rangle, \langle v, \{\emptyset\}\rangle\} = \{u\} \times \{\emptyset\} \cup \{v\} \times \{\{\emptyset\}\}$.

(32) Let us consider elements I, a, b, u, v of $\mathcal{U}$, and a $\mathcal{U}$-valued many sorted set x indexed by I. Suppose $I = \{a, b\}$ and $x(a) = u$ and $x(b) = v$. Then $\biguplus x = u \times \{a\} \cup v \times \{b\}$.

Let us consider elements I, u, v of $\mathcal{U}$ and a $\mathcal{U}$-valued many sorted set x indexed by I. Now we state the propositions:

(33) Suppose $I = \{\emptyset, \{\emptyset\}\}$ and $x(\emptyset) = u$ and $x(\{\emptyset\}) = v$. Then $\biguplus x = u \times \{\emptyset\} \cup v \times \{\{\emptyset\}\}$. The theorem is a consequence of (32).

(34) Suppose $I = \{\emptyset, \{\emptyset\}\}$ and $x(\emptyset) = \{u\}$ and $x(\{\emptyset\}) = \{v\}$ and $u \neq v$. Then $\biguplus x = u \uplus v$. The theorem is a consequence of (33) and (31).

(35) Let us consider an element x of $\mathcal{U}$, and objects y, z. Suppose $x = \langle y, z\rangle$. Then

(i) y is an element of $\mathcal{U}$, and

(ii) z is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class. Observe that there exists an element of $\mathcal{U}$ which is pair.

Let u be a pair element of $\mathcal{U}$. The functors: $(u)_{\mathbf{1}}$ and $(u)_{\mathbf{2}}$ yield elements of $\mathcal{U}$. Now we state the proposition:

(36) Let us consider an element X of $\mathcal{U}$. Then

(i) $\pi_1(X)$ is an element of $\mathcal{U}$, and

(ii) $\pi_2(X)$ is an element of $\mathcal{U}$.

The theorem is a consequence of (1).

Let us consider a binary relation R. Now we state the propositions:

(37) If $R \in \mathcal{U}$, then $\mathrm{dom}\, R$, $\mathrm{rng}\, R \in \mathcal{U}$. The theorem is a consequence of (36).

(38) If $\mathrm{dom}\, R$ is an element of $\mathcal{U}$ and $\mathrm{rng}\, R$ is an element of $\mathcal{U}$, then R is an element of $\mathcal{U}$. The theorem is a consequence of (13).

(39) Let us consider a set X, a non empty set Y, and a function f from X into Y. If $f \in \mathcal{U}$, then $X \in \mathcal{U}$. The theorem is a consequence of (37).

(40) Let us consider non empty sets A, B. Suppose $A \times B$ is an element of $\mathcal{U}$. Then

(i) A is an element of $\mathcal{U}$, and

(ii) B is an element of $\mathcal{U}$.

The theorem is a consequence of (36).

(41) Let us consider a set X. Suppose id_X is an element of $\mathcal{U}$. Then X is an element of $\mathcal{U}$. The theorem is a consequence of (37).

(42) Let us consider elements x, y, z of $\mathcal{U}$. Then $\langle x, y \rangle \longmapsto z$ is an element of $\mathcal{U}$.

6. PROPERTIES OF UNIVERSE CONTAINING ω

Now we state the propositions:

(43) $\omega \subset \mathbf{U}_0$. The theorem is a consequence of (16).

(44) Let us consider a set X. Then $\mathbf{T}(\emptyset) \subseteq \mathbf{T}(X)$.

(45) Let us consider a Grothendieck $\mathcal{G}$ of X. Then $\mathbf{U}_0 \subseteq \mathcal{G}$. The theorem is a consequence of (44).

(46) (i) GrothendieckUniverse($\emptyset$) $= \mathbf{U}_0$, and

(ii) GrothendieckUniverse($\emptyset$) $= \mathbf{U}_\emptyset$.

(47) Let us consider a set X, and a Grothendieck $\mathcal{G}$ of X. Then Grothendieck Universe($\emptyset$) $\subseteq$ GrothendieckUniverse(X) $\subseteq \mathcal{G}$.

(48) Let us consider an element n of $\mathbf{U}_0$. Then GrothendieckUniverse(n) $=$ $\mathbf{U}_0$. The theorem is a consequence of (45).

(49) the empty Grothendieck $\subset \omega \subset$ GrothendieckUniverse($\emptyset$) $\subset$ Grothendieck Universe(ω). The theorem is a consequence of (16), (46), (43), (19), and (20).

(50) Let us consider a non empty Grothendieck $\mathcal{G}$. Suppose $\mathcal{G} \neq$ Grothendieck Universe(ω). Then

(i) GrothendieckUniverse(ω) $\in \mathcal{G}$, or

(ii) $\mathcal{G} \in$ GrothendieckUniverse(ω).

(51) $\mathbf{T}(\omega) =$ GrothendieckUniverse(ω).

(52) Let us consider sets N_1, N_2. Suppose $N_1 = \mathbb{N} \times \mathbb{N} \cup \mathbb{N}$ and $N_2 = N_1 \cup 2^{N_1}$. Then $\mathbb{R} \subseteq N_2 \cup \mathbb{N} \times N_2$.

Let us consider a non trivial universal class $\mathcal{U}$. Now we state the propositions:

(53) $\mathbb{R}$ is an element of $\mathcal{U}$. The theorem is a consequence of (52) and (13).

(54) $\overline{\mathbb{R}}$ is an element of $\mathcal{U}$. The theorem is a consequence of (53) and (13).

(55) $\mathbb{C} \in \mathcal{U}$. The theorem is a consequence of (16), (53), and (13).

(56) $\mathbb{H} \in \mathcal{U}$. The theorem is a consequence of (16), (53), (55), and (13).

(57) Let us consider a natural number n. Then $\mathrm{Seg}\, n \in \mathcal{U}$. The theorem is a consequence of (16) and (13).

(58) Let us consider a set D. If $D \in \mathcal{U}$, then for every natural number n, $D^n \in \mathcal{U}$. The theorem is a consequence of (57).

(59) Let us consider a non trivial universal class $\mathcal{U}$, and a natural number n. Then $\mathcal{R}^n \in \mathcal{U}$. The theorem is a consequence of (53) and (58).

Let us consider a set X and a natural number n. Now we state the propositions:

(60) If $X \in \mathcal{U}$, then $X^n \in \mathcal{U}$. The theorem is a consequence of (57).

(61) $X^n \subseteq X^*$.

(62) Let us consider a non empty set X, and an object x. If $x \in X^*$, then there exists a natural number n such that $x \in X^n$.

(63) Let us consider a non empty set X. Then there exists a function f such that

  (i) $\operatorname{dom} f = \mathbb{N}$, and

  (ii) for every natural number n, $f(n) = X^n$, and

  (iii) $\bigcup \operatorname{rng} f = X^*$.

  Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number n such that $\$_1 = n$ and $\$_2 = X^n$. For every object x such that $x \in \mathbb{N}$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and for every object x such that $x \in \mathbb{N}$ holds $\mathcal{P}[x, f(x)]$. For every natural number n, $f(n) = X^n$. $\bigcup \operatorname{rng} f = X^*$. □

(64) Let us consider a non trivial universal class $\mathcal{U}$, and a non empty set X. If $X \in \mathcal{U}$, then $X^* \in \mathcal{U}$. The theorem is a consequence of (63) and (58).

Let us consider a non trivial universal class $\mathcal{U}$. Now we state the propositions:

(65) $\mathbb{R}^* \in \mathcal{U}$. The theorem is a consequence of (53) and (64).

(66) $\overline{\mathbb{R}}^* \in \mathcal{U}$. The theorem is a consequence of (54) and (64).

(67) $\mathbb{C}^* \in \mathcal{U}$.

(68) $(\mathbb{H})^* \in \mathcal{U}$.

(69) Let us consider a universal class $\mathcal{U}$, and a set X. If $X \in \mathcal{U}$, then for every finite sequence s of elements of X, $s \in \mathcal{U}$. The theorem is a consequence of (57) and (13).

(70) Let us consider an empty set X, and a finite sequence f of elements of X^*. Then $f = \operatorname{len} f \mapsto 0$.

(71) Let us consider a non trivial universal class $\mathcal{U}$, and a non empty set D. If $D \in \mathcal{U}$, then for every matrix M over D, $M \in \mathcal{U}$.

(72) $\mathbf{U}_0, \mathbb{N}, \mathbb{R}, \overline{\mathbb{R}} \in \mathbf{U}_1$. The theorem is a consequence of (16), (13), (53), and (54).

(73) Let us consider a set X, and a universal class $\mathcal{U}$. If $\mathcal{U} \in \mathbf{T}(X)$, then $\mathbf{T}(\mathcal{U}) \subseteq \mathbf{T}(X)$.

(74) $\mathbf{U}_0 \in \mathbf{T}(\omega)$. The theorem is a consequence of (19) and (20).

(75) $\mathbf{U}_1 = \mathbf{T}(\omega)$. The theorem is a consequence of (72), (73), and (74).

(76) $\mathrm{GrothendieckUniverse}(\omega) = \mathbf{U}_1$.

(77) $\mathrm{GrothendieckUniverse}(\omega) = \mathrm{GrothendieckUniverse}(\mathbf{U}_0) = \mathbf{U}_1$.
Proof: $\mathrm{GrothendieckUniverse}(\omega) = \mathrm{GrothendieckUniverse}(\mathbf{U}_0)$. □

Let us consider a non empty set X, a Grothendieck $\mathcal{G}'$ of X, and a universal class $\mathcal{G}$. Now we state the propositions:

(78) If X misses $\mathcal{G}$, then $\mathcal{G}' \neq \mathcal{G}$.

(79) If X misses $\mathcal{G}$, then $\mathcal{G}' \in \mathcal{G}$ or $\mathcal{G} \in \mathcal{G}'$.

(80) Let us consider universal classes $\mathcal{U}, \mathcal{U}'$, and an element a of $\mathcal{U}$. If $a \notin \mathcal{U}'$, then $\mathcal{U}' \in \mathcal{U}$. The theorem is a consequence of (78).

(81) Let us consider a Grothendieck $\mathcal{G}$. Then $\bigcup \mathcal{G} = \mathcal{G}$.

One can verify that every Grothendieck is limit ordinal.

Now we state the proposition:

(82) Let us consider a universal class $\mathcal{U}$, and a non empty element V of $\mathcal{U}$. Then $\mathrm{Funcs}\, V$ is a subset of $\mathcal{U}$. The theorem is a consequence of (81).

7. How to Get Out of a Universe?

Now we state the propositions:

(83) There exists a set a such that $a \notin \mathcal{U}$.

(84) There exists a subset A of $\mathcal{U}$ such that $A \notin \mathcal{U}$.

(85) the set of all u where u is an element of $\mathcal{U}$ is not an element of $\mathcal{U}$.

(86) Let us consider an element X of $\mathcal{U}$. Then $\mathcal{U} \setminus X$ is not an element of $\mathcal{U}$.
Proof: $\mathcal{U} \setminus X \notin \mathcal{U}$. □

(87) $2^{\mathcal{U}} \notin \mathcal{U}$.

8. A Sequence of Universes

Now we state the proposition:

(88) Let us consider a set X. Then there exists a function f such that

(i) $\mathrm{dom}\, f = \mathbb{N}$, and

(ii) $f(0) = X$, and

(iii) for every natural number n, $f(n+1) = \mathrm{GrothendieckUniverse}(f(n))$.

Proof: Define $\mathcal{G}(\text{set}, \text{set}) = \text{GrothendieckUniverse}(\$_2)$. There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = X$ and for every natural number n, $f(n+1) = \mathcal{G}(n, f(n))$. □

The Construction of X, GrothendieckUniverse(X), GrothendieckUniverse (GrothendieckUniverse(X)), ...

Let X be a set. The functor sequence-universe(X) yielding a function is defined by

(Def. 9) dom $it = \mathbb{N}$ and $it(0) = X$ and for every natural number n, $it(n+1) =$ GrothendieckUniverse($it(n)$).

Now we state the propositions:

(89) Let us consider a set X. Then sequence-universe(X) is a transfinite sequence.

(90) Let us consider a set X, and a transfinite sequence S. If dom $S = \mathbb{N}$, then last $S = S(\mathbb{N})$.

(91) Let us consider a transfinite sequence S. Suppose dom $S = \mathbb{N}$. Then

(i) $S(\mathbb{N}) = \emptyset$, and

(ii) last $S = \emptyset$.

The theorem is a consequence of (90).

(92) Let us consider a set X, and a transfinite sequence S. Suppose $S =$ sequence-universe(X). Then

(i) last $S = \emptyset$, and

(ii) $S(\mathbb{N}) = \emptyset$.

The theorem is a consequence of (91).

The Construction of $X \cup$GrothendieckUniverse(X)$\cup$GrothendieckUniverse(GrothendieckUniverse(X)) $\cup \ldots$.

Let X be a set. The functor union-sequence-universe(X) yielding a non empty set is defined by the term

(Def. 10) $\bigcup$ rng sequence-universe(X).

Now we state the proposition:

(93) Let us consider a set X. Then rng sequence-universe(X) $\subseteq$ union-sequence-universe(X).

The Formal Counterpart of $\emptyset(=\mathcal{U}_0) \in \mathcal{U}_1 \in \mathcal{U}_2 \in \ldots$: Sequence of universes indexed by the ordinal numbers (see 8. Examples, Grothendieck Universe [24]).

The functor sequence-universe yielding a sequence of union-sequence-universe($\emptyset$) is defined by the term

(Def. 11) sequence-universe($\emptyset$).

Now we state the propositions:

(94) $\emptyset, \mathbf{U}_0, \mathbf{U}_1 \in \operatorname{rng}$ sequence-universe. The theorem is a consequence of (45) and (77).

(95) $\bigcup_{n<\omega} \mathcal{U}_n$ is not a Universe:
$\bigcup \operatorname{rng}$ sequence-universe is not a Grothendieck. The theorem is a consequence of (72) and (94).

(96) (i) $\mathbf{T}(\mathbf{U}_0) = \text{GrothendieckUniverse}(\mathbf{U}_0)$, and

(ii) $\mathbf{T}(\mathbf{U}_1) = \text{GrothendieckUniverse}(\mathbf{U}_1)$.

(97) Let us consider a set X, and a natural number n. Then

(i) $(\text{sequence-universe}(X))(n+1)$ is transitive, and

(ii) $\mathbf{T}((\text{sequence-universe}(X))(n+1)) =$
$\text{GrothendieckUniverse}((\text{sequence-universe}(X))(n+1))$.

Let us consider a natural number n. Now we state the propositions:

(98) $\mathbf{T}((\text{sequence-universe}(\mathbf{U}_0))(n)) =$
$\text{GrothendieckUniverse}((\text{sequence-universe}(\mathbf{U}_0))(n))$. The theorem is a consequence of (77).

(99) $\mathbf{U}_n \in \mathbf{U}_{n+1}$.

(100) $(\text{sequence-universe}(\mathbf{U}_0))(n) = \mathbf{U}_n$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\text{sequence-universe}(\mathbf{U}_0))(\$_1) = \mathbf{U}_{\$_1}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(101) $\text{GrothendieckUniverse}((\text{sequence-universe}(\emptyset))(n)) =$
$(\text{sequence-universe}(\text{GrothendieckUniverse}(\emptyset)))(n)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv \text{GrothendieckUniverse}((\text{sequence-universe}(\emptyset))(\$_1)) = (\text{sequence-universe}(\text{GrothendieckUniverse}(\emptyset)))(\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(102) $(\text{sequence-universe})(n+1) = \mathbf{U}_n$. The theorem is a consequence of (46), (100), and (101).

Let us note that there exists an element of $\bigcup \operatorname{rng}$ sequence-universe which is non empty.

Now we state the propositions:

(103) $\mathbf{U}_0, \mathbf{U}_1 \in \text{GrothendieckUniverse}(\text{sequence-universe})$. The theorem is a consequence of (45) and (77).

(104) Let us consider a natural number n. Then $(\text{sequence-universe})(n+1) \in \text{GrothendieckUniverse}(\text{sequence-universe})$. The theorem is a consequence of (45) and (102).

The Construction of $\mathcal{U}_\omega$: Tower of universes indexed by the ordinal numbers (see 8. Examples, Grothendieck Universe [24]).

The functor $\mathcal{U}_\omega$ yielding a non trivial universal class is defined by the term

(Def. 12) GrothendieckUniverse(sequence-universe).

Now we state the proposition:

(105) Let us consider a natural number n. Then (sequence-universe)$(n) \subseteq$ (sequence-universe)$(n+1)$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (sequence-universe)$(\$_1) \subseteq$ (sequence-universe)$(\$_1+1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. □

Let X be an element of $\bigcup$ rng sequence-universe. The functor rank-universe(X) yielding a natural number is defined by

(Def. 13) $X \in$ (sequence-universe)(it) and for every natural number n such that $n < it$ holds $X \notin$ (sequence-universe)(n).

Now we state the propositions:

(106) Let us consider an element X of $\bigcup$ rng sequence-universe, and a natural number n. Suppose rank-universe$(X) \leqslant n$.
Then $X \in$ (sequence-universe)(n).
PROOF: Define $\mathcal{P}$[natural number] $\equiv X \in$ (sequence-universe)$(\$_1)$. For every natural number j such that rank-universe$(X) \leqslant j$ and $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$. For every natural number i such that rank-universe$(X) \leqslant i$ holds $\mathcal{P}[i]$. □

(107) Let us consider a natural number i. Then there exists a set x such that $x \in$ (sequence-universe)$(i+1) \setminus$ (sequence-universe)(i). The theorem is a consequence of (105) and (102).

(108) Let us consider a natural number n. Then $\mathbf{U}_{n+1} \setminus (\mathbf{U}_n) \notin \mathbf{U}_{n+1}$. The theorem is a consequence of (99) and (86).

The functor ComplUniverse yielding a function from $\mathbb{N}$ into $\bigcup$ rng sequence-universe is defined by

(Def. 14) for every natural number n, $it(n) = \mathbf{U}_{n+1} \setminus (\mathbf{U}_n)$.

Let us consider a natural number n. Now we state the propositions:

(109) (ComplUniverse)(n) is not empty. The theorem is a consequence of (99).

(110) (ComplUniverse)$(n) \subseteq \mathbf{U}_{n+1}$.

(111) There exists a function f from $\mathbb{N}$ into $\bigcup\bigcup$ rng sequence-universe such that for every natural number i, $f(i) \in$ (ComplUniverse)(i).
PROOF: Set $g =$ the choice of ComplUniverse. rng $g \subseteq \bigcup\bigcup$ rng sequence-universe. For every natural number i, $g(i) \in$ (ComplUniverse)(i). □

(112) Let us consider a function f from $\mathbb{N}$ into $\bigcup \operatorname{rng}$ sequence-universe. Then $f \in \mathcal{U}_\omega$. The theorem is a consequence of (13) and (104).

(113) Let us consider a function f from $\mathbb{N}$ into $\bigcup\bigcup \operatorname{rng}$ sequence-universe. Then $f \in \mathcal{U}_\omega$. The theorem is a consequence of (13) and (104).

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