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Compactness of Neural Networks[1]

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Summary. In this article, Feed-forward Neural Network is formalized in the Mizar system [1], [2]. First, the multilayer perceptron [6], [7], [8] is formalized using functional sequences. Next, we show that a set of functions generated by these neural networks satisfies equicontinuousness and equiboundedness property [10], [5]. At last, we formalized the compactness of the function set of these neural networks by using the Ascoli-Arzela's theorem according to [4] and [3].

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1. Preliminaries

From now on R_1, R_2 denote real linear spaces.

Now we state the propositions:

(1) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then the carrier of R_1 = the carrier of R_2.

(2) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then $0_{R_1} = 0_{R_2}$.

(3) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider elements p, q of R_1, and elements f, g of R_2. If $p = f$ and $q = g$, then $p + q = f + g$.

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(4) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a real number r, an element q of R_1, and an element g of R_2. If $q = g$, then $r \cdot q = r \cdot g$.

(5) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider an element q of R_1, and an element g of R_2. If $q = g$, then $-q = -g$.

(6) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider elements p, q of R_1, and elements f, g of R_2. If $p = f$ and $q = g$, then $p - q = f - g$.

(7) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a set X, and a natural number n. Then X is a linear combination of R_2 if and only if X is a linear combination of R_1.

(8) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L_5 of R_1, and a linear combination L_3 of R_2. Suppose $L_3 = L_5$. Then the support of L_3 = the support of L_5.

Let us consider a set F. Now we state the propositions:

(9) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a subset of R_1 if and only if F is a subset of R_2.

(10) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a finite sequence of elements of R_1 if and only if F is a finite sequence of elements of R_2.

(11) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then F is a function from R_1 into $\mathbb{R}$ if and only if F is a function from R_2 into $\mathbb{R}$.

(12) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a finite sequence F_1 of elements of R_1, a function f_1 from R_1 into $\mathbb{R}$, a finite sequence F_3 of elements of R_2, and a function f_2 from R_2 into $\mathbb{R}$. If $f_1 = f_2$ and $F_1 = F_3$, then $f_1 \cdot F_1 = f_2 \cdot F_3$.

(13) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a finite sequence F_2 of elements of R_1, and a finite sequence F_1 of elements of R_2. If $F_2 = F_1$, then $\sum F_2 = \sum F_1$.
PROOF: Set $T = R_1$. Set $V = R_2$. Consider f being a sequence of the carrier of T such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_T$ and for every natural number j and for every element v of T such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j)+v$. Consider f_2 being a sequence of the carrier of V such that $\sum F_3 = f_2(\operatorname{len} F_3)$ and $f_2(0) = 0_V$ and for every natural number j and for every element v of V such that $j < \operatorname{len} F_3$ and $v = F_3(j+1)$ holds $f_2(j+1) = f_2(j)+v$. Define $\mathcal{S}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $f(\$_1) = f_2(\$_1)$. For every natural number i such that $\mathcal{S}[i]$ holds $\mathcal{S}[i+1]$. For every natural number n, $\mathcal{S}[n]$. $\square$

(14) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L_3 of R_2, and a linear combination L_4 of R_1. If $L_3 = L_4$, then $\sum L_3 = \sum L_4$. The theorem is a consequence of (12) and (13).

(15) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a subset A_1 of R_2, and a subset A_2 of R_1. Suppose $A_1 = A_2$. Let us consider an object X. Then X is a linear combination of A_1 if and only if X is a linear combination of A_2. The theorem is a consequence of (7).

Let us consider a subset A_1 of R_2 and a subset A_2 of R_1. Now we state the propositions:

(16) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then if $A_1 = A_2$, then $\Omega_{\mathrm{Lin}(A_1)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (7) and (14).

(17) Suppose the RLS structure of R_1 = the RLS structure of R_2. Then if $A_1 = A_2$, then A_1 is linearly independent iff A_2 is linearly independent. The theorem is a consequence of (7) and (14).

(18) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider an object X. Then X is a subspace of R_2 if and only if X is a subspace of R_1.

(19) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a linear combination L of R_2, and a linear combination S of R_1. If $L = S$, then $\sum L = \sum S$. The theorem is a consequence of (12) and (13).

(20) Suppose the RLS structure of R_1 = the RLS structure of R_2. Let us consider a set X. Then X is a basis of R_1 if and only if X is a basis of R_2. The theorem is a consequence of (17) and (16).

(21) Let us consider real linear spaces R_1, R_2. Suppose the RLS structure of R_1 = the RLS structure of R_2 and R_1 is finite dimensional. Then

   (i) R_2 is finite dimensional, and

   (ii) $\dim(R_2) = \dim(R_1)$.

The theorem is a consequence of (20).

Let us consider a real normed space R_3. Now we state the propositions:

(22) The normed structure of R_3 is a strict real normed space.

(23) There exists a normed linear topological space T such that the normed structure of R_3 = the normed structure of T.

Proof: Reconsider R_3 = the normed structure of $RNS0$ as a strict real normed space. Set L_2 = LinearTopSpaceNorm R_3. Reconsider N = the norm of R_3 as a function from the carrier of L_2 into $\mathbb{R}$. Set W =

$\langle$the carrier of L_2, the zero of L_2, the addition of L_2, the external multiplication of L_2, the topology of $L_2, N\rangle$. W is topological space-like, right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, add-continuous, and mult-continuous. □

(24) Suppose R_3 is finite dimensional. Then there exists a normed linear topological space T such that

(i) the normed structure of $R_3 =$ the normed structure of T, and

(ii) T is finite dimensional.

The theorem is a consequence of (23) and (21).

(25) Let us consider a normed linear topological space T, and a real normed space R_3. Suppose T is finite dimensional and $R_3 =$ the normed structure of T. Then

(i) R_3 is finite dimensional, and

(ii) $\dim(R_3) = \dim(T)$.

The theorem is a consequence of (21).

2. The Ascoli-Arzela Theorem on Finite Dimensional Normed Linear Spaces

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of T)$^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T).

Now we state the propositions:

(26) Suppose $S = M_{\rm top}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Then suppose $G = H$. Then MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded if and only if G is equibounded and equicontinuous.

Proof: For every point x of S and for every non empty subset H_1 of MetricSpaceNorm T such that $H_1 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T\restriction\overline{H_1}$ is compact by [9, (1)], (25). □

(27) Suppose $S = M_{\rm top}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Then if $G = H$, then $\overline{H}$ is sequentially compact iff G is equibounded and equicontinuous. The theorem is a consequence of (26).

(28) Let us consider a non empty metric space M, a non empty, compact topological space S, and a normed linear topological space T. Suppose $S = M_{\text{top}}$ and T is complete and finite dimensional and $\dim(T) \neq 0$. Let us consider a subset G of (the carrier of $T)^{\alpha}$, and a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T. Suppose $G = F$. Then $\overline{F}$ is compact if and only if G is equibounded and equicontinuous, where α is the carrier of M. The theorem is a consequence of (27).

(29) Let us consider a non empty real normed space R_3, a normed linear topological space T, a non empty subset X of R_3, a non empty, compact, strict topological space S, and a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T.

Suppose S is a subspace of TopSpaceNorm R_3 and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and there exist real numbers K, D such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of R_3 such that $x, y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of R_3 such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. Then $\overline{G}$ is compact.

Proof: Reconsider $Y = X$ as a non empty subset of MetricSpaceNorm R_3. Reconsider $M =$ MetricSpaceNorm $R_3 \restriction Y$ as a non empty metric space. For every object z, $z \in$ the topology of S iff $z \in$ the open set family of M. For every object z such that $z \in$ the continuous functions of S and T holds $z \in$ (the carrier of $T)^{\alpha}$, where α is the carrier of M. Reconsider $H = G$ as a subset of (the carrier of $T)^{(\text{the carrier of } M)}$. $\overline{G}$ is compact iff H is equibounded and equicontinuous.

Consider K, D being real numbers such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of R_3 such that $x, y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of R_3 such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. For every function f from the carrier of M into the carrier of T such that $f \in H$ for every element x of M, $\|f(x)\| \leqslant K$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in H$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

3. High-Order and Multilayer Perceptron

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. We say that N is a multilayer perceptron with k and n if and only if

(Def. 1) $\operatorname{len} N = n$ and $\operatorname{len} N+1 = \operatorname{len} k$ and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

We say that N is a multilayer perceptron-like if and only if

(Def. 2) there exists a finite sequence k of elements of $\mathbb{N}$ such that $\operatorname{len} N+1 = \operatorname{len} k$ and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

Observe that there exists a finite sequence which is a multilayer perceptron-like. A multilayer perceptron is multilayer perceptron-like finite sequence. Now we state the proposition:

(30) Let us consider a multilayer perceptron N. Then there exists a finite sequence k of elements of $\mathbb{N}$ such that

(i) $\operatorname{len} N + 1 = \operatorname{len} k$, and

(ii) for every natural number i such that $1 \leqslant i < \operatorname{len} k$ holds $N(i)$ is a function from $\langle \mathcal{E}^{k(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$.

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. Assume N is a multilayer perceptron with k and n. Assume $\operatorname{len} N \neq 0$. The functor $\operatorname{OutputFunc}(N, k, n)$ yielding a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ is defined by

(Def. 3) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} N$ and $p(1) = N(1)$ and for every natural number i such that $1 \leqslant i < \operatorname{len} N$ there exists a function N_2 from $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+2)}, \|\cdot\| \rangle$ and there exists a function p_2 from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ such that $N_2 = N(i+1)$ and $p_2 = p(i)$ and $p(i+1) = N_2 \cdot p_2$ and $it = p(\operatorname{len} N)$.

Now we state the proposition:

(31) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, and a non empty finite sequence N. Suppose $n \neq 0$ and N is a multilayer perceptron with k and $n+1$. Then there exists a finite sequence k_1 of elements of $\mathbb{N}$ and there exists a non empty finite sequence N_1 and there exists a function N_2 from $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+2)}, \|\cdot\| \rangle$ such that $N_1 = N{\restriction}n$ and $k_1 = k{\restriction}(n+1)$ and $N_2 = N(n+1)$ and N_1 is a multilayer perceptron with k_1 and n and $\operatorname{OutputFunc}(N, k, n+1) = N_2 \cdot (\operatorname{OutputFunc}(N_1, k_1, n))$.

PROOF: Reconsider $N_1 = N \restriction n$ as a non empty finite sequence. Reconsider $k_1 = k \restriction (n+1)$ as a finite sequence of elements of $\mathbb{N}$. For every natural number i such that $1 \leqslant i < \operatorname{len} k_1$ holds $N_1(i)$ is a function from $\langle \mathcal{E}^{k_1(i)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k_1(i+1)}, \|\cdot\| \rangle$. Consider p being a finite sequence such that $\operatorname{len} p = \operatorname{len} N$ and $p(1) = N(1)$ and for every natural number i such that $1 \leqslant i < \operatorname{len} N$ there exists a function N_2 from $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+2)}, \|\cdot\| \rangle$ and there exists a function p_2 from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \|\cdot\| \rangle$ such that $N_2 = N(i+1)$ and $p_2 = p(i)$ and $p(i+1) = N_2 \cdot p_2$ and $\mathrm{OutputFunc}(N, k, n+1) = p(\operatorname{len} N)$. Consider N_2 being a function from $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+2)}, \|\cdot\| \rangle$, p_2 being a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$ such that $N_2 = N(n+1)$ and $p_2 = p(n)$ and $p(n+1) = N_2 \cdot p_2$. $\square$

Let n be a natural number and k be a finite sequence of elements of $\mathbb{N}$. The functor $\mathrm{Neurons}(n, k)$ yielding a subset of

(the carrier of $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle)^{(\text{the carrier of } \langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle)}$ is defined by the term

(Def. 4) $\{F$, where F is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle$: there exists a finite sequence N such that N is a multilayer perceptron with k and n and $F = \mathrm{OutputFunc}(N, k, n)\}$.

Now we state the propositions:

(32) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subspace M of $\mathrm{MetricSpaceNorm}\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, and a normed linear topological space T. Suppose $S = M_{\mathrm{top}}$ and the carrier of $M = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle =$ the normed structure of T.

Let us consider a subset G of (the carrier of $T)^{\alpha}$, and a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T. Suppose $G = F$ and $G \subseteq \{f \restriction X$, where f is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle : f \in \mathrm{Neurons}(n, k)\}$. Then $\overline{F}$ is compact if and only if G is equibounded and equicontinuous, where α is the carrier of M.

(33) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$, and a normed linear topological space T. Suppose S is a subspace of $\mathrm{TopSpaceNorm}\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle =$ the normed structure of T. Let us consider a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T.

Suppose $G \subseteq \{f \restriction X$, where f is a function from $\langle \mathcal{E}^{k(1)}, \|\cdot\| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \|\cdot\| \rangle : f \in \mathrm{Neurons}(n, k)\}$ and there exist real numbers K, D

such that $0 < K$ and $0 < D$ and for every function F from X into T such that $F \in G$ holds for every points x, y of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x, y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D \cdot \|x - y\|$ and for every point x of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x \in X$ holds $\|F_{/x}\| \leqslant K$. Then $\overline{G}$ is compact.

Let X, Y be real normed spaces, F be a function from X into Y, and D, K be real numbers. We say that F is a layer function of D and K if and only if

(Def. 5) for every points x, y of X, $\|F(x) - F(y)\| \leqslant D \cdot \|x - y\|$ and for every point x of X, $\|F(x)\| \leqslant K$.

Let n be a natural number, k be a finite sequence of elements of $\mathbb{N}$, and N be a finite sequence. We say that N is a layer sequence of D, K, k and n if and only if

(Def. 6) $\operatorname{len} N = n$ and N is a multilayer perceptron with k and n and for every natural number i such that $1 \leqslant i < \operatorname{len} k$ there exists a function N_3 from $\langle \mathcal{E}^{k(i)}, \| \cdot \| \rangle$ into $\langle \mathcal{E}^{k(i+1)}, \| \cdot \| \rangle$ such that $N(i) = N_3$ and N_3 is a layer function of D and K.

Now we state the propositions:

(34) Let us consider real numbers D, K. Suppose $0 \leqslant D$ and $0 \leqslant K$. Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, and a non empty finite sequence N. Suppose N is a layer sequence of D, K, k and n. Then $\operatorname{OutputFunc}(N, k, n)$ is a layer function of D^n and K.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence k of elements of $\mathbb{N}$ for every non empty finite sequence N such that $\operatorname{len} N = \$_1$ and N is a layer sequence of D, K, k and $\$_1$ holds $\operatorname{OutputFunc}(N, k, \$_1)$ is a layer function of $D^{\$_1}$ and K. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(35) Let us consider a natural number n, a finite sequence k of elements of $\mathbb{N}$, a non empty, compact, strict topological space S, a non empty subset X of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$, and a normed linear topological space T. Suppose S is a subspace of $\operatorname{TopSpaceNorm}\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ and the carrier of $S = X$ and X is compact and T is complete and finite dimensional and $\dim(T) \neq 0$ and $\langle \mathcal{E}^{k(n+1)}, \| \cdot \| \rangle =$ the normed structure of T.

Let us consider a non empty subset G of the $\mathbb{R}$-norm space of continuous functions of S and T, and real numbers D, K. Suppose $0 < D$ and $0 < K$ and $G \subseteq \{F \restriction X$, where F is a function from $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ into $\langle \mathcal{E}^{k(n+1)}, \| \cdot \| \rangle$: there exists a non empty finite sequence N such that N is a layer sequence of D, K, k and n and $F = \operatorname{OutputFunc}(N, k, n)\}$. Then $\overline{G}$ is compact.

PROOF: Set $K_1 = K + 1$. Set $D_1 = D^n + 1$. For every function F from X into T such that $F \in G$ holds for every points x, y of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that x, $y \in X$ holds $\|F_{/x} - F_{/y}\| \leqslant D_1 \cdot \|x - y\|$ and for every point x of $\langle \mathcal{E}^{k(1)}, \| \cdot \| \rangle$ such that $x \in X$ holds $\|F_{/x}\| \leqslant K_1$. □

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