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Intuitionistic Propositional Calculus in the Extended Framework with Modal Operator. Part II

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Summary. This paper is a continuation of Inoué [5]. As already mentioned in the paper, a number of intuitionistic provable formulas are given with a Hilbert-style proof. For that, we make use of a family of intuitionistic deduction theorems, which are also presented in this paper by means of Mizar system [2], [1]. Our axiom system of intuitionistic propositional logic IPC is based on the propositional subsystem of H_1-**IQC** in Troelstra and van Dalen [6, p. 68]. We also owe Heyting [4] and van Dalen [7]. Our treatment of a set-theoretic intuitionistic deduction theorem is due to Agata Darmochwał's Mizar article "Calculus of Quantifiers. Deduction Theorem" [3].

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1. The Notion of Proof in Intuitionistic Setting

From now on i, j, n, k, l denote natural numbers, T, S, X, Y, Z denote subsets of MC-w.f.f., p, q, r, t, F, H, G denote elements of MC-w.f.f., and s, U, V denote MC-formulas.

Let p, q be elements of MC-w.f.f.. The functor $p \Leftrightarrow q$ yielding an element of MC-w.f.f. is defined by the term

(Def. 1) $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

The functor Proof-Step-Kinds-IPC yielding a set is defined by the term

(Def. 2) $\{k : k \leqslant 10\}$.

Now we state the proposition:

(1) (i) $0 \in$ Proof-Step-Kinds-IPC and ... and

(ii) $10 \in$ Proof-Step-Kinds-IPC.

One can verify that Proof-Step-Kinds-IPC is non empty and Proof-Step-Kinds-IPC is finite.

From now on f, g denote finite sequences of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC. Now we state the proposition:

(2) Let us consider a natural number n. If $1 \leqslant n \leqslant \operatorname{len} f$, then $(f(n))_{\mathbf{2}} = 0$ or ... or $(f(n))_{\mathbf{2}} = 10$.

Let P_1 be a finite sequence of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC and n be a natural number. Let us consider X. We say that P_1 is a correct n-th step w.r.t. IPC (X) if and only if

(Def. 3) (i) $(P_1(n))_{\mathbf{1}} \in X$, **if** $(P_1(n))_{\mathbf{2}} = 0$,

(ii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow p)$, **if** $(P_1(n))_{\mathbf{2}} = 1$,

(iii) there exists p and there exists q and there exists r such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$, **if** $(P_1(n))_{\mathbf{2}} = 2$,

(iv) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \wedge q \Rightarrow p$, **if** $(P_1(n))_{\mathbf{2}} = 3$,

(v) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \wedge q \Rightarrow q$, **if** $(P_1(n))_{\mathbf{2}} = 4$,

(vi) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow (q \Rightarrow p \wedge q)$, **if** $(P_1(n))_{\mathbf{2}} = 5$,

(vii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow p \vee q$, **if** $(P_1(n))_{\mathbf{2}} = 6$,

(viii) there exists p and there exists q such that $(P_1(n))_{\mathbf{1}} = q \Rightarrow p \vee q$, **if** $(P_1(n))_{\mathbf{2}} = 7$,

(ix) there exists p and there exists q and there exists r such that $(P_1(n))_{\mathbf{1}} = p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$, **if** $(P_1(n))_{\mathbf{2}} = 8$,

(x) there exists p such that $(P_1(n))_{\mathbf{1}} = \mathrm{FALSUM} \Rightarrow p$, **if** $(P_1(n))_{\mathbf{2}} = 9$,

(xi) there exists i and there exists j and there exists p and there exists q such that $1 \leqslant i < n$ and $1 \leqslant j < i$ and $p = (P_1(j))_{\mathbf{1}}$ and $q = (P_1(n))_{\mathbf{1}}$ and $(P_1(i))_{\mathbf{1}} = p \Rightarrow q$, **if** $(P_1(n))_{\mathbf{2}} = 10$.

Let us consider f. We say that f is a proof w.r.t. IPC (X) if and only if

(Def. 4) $f \neq \emptyset$ and for every n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds f is a correct n-th step w.r.t. IPC (X).

Now we state the propositions:

(3) If f is a proof w.r.t. IPC (X), then $\operatorname{rng} f \neq \emptyset$.

(4) If f is a proof w.r.t. IPC (X), then $1 \leqslant \operatorname{len} f$.

(5) If f is a proof w.r.t. IPC (X), then $(f(1))_{\mathbf{2}} = 0$ or ... or $(f(1))_{\mathbf{2}} = 10$. The theorem is a consequence of (4) and (2).

(6) If $1 \leqslant n \leqslant \operatorname{len} f$, then f is a correct n-th step w.r.t. IPC (X) iff $f \frown g$ is a correct n-th step w.r.t. IPC (X).

PROOF: If f is a correct n-th step w.r.t. IPC (X), then $f \frown g$ is a correct n-th step w.r.t. IPC (X). $(f(n))_{\mathbf{2}} = 0$ or ... or $(f(n))_{\mathbf{2}} = 10$. □

(7) If $1 \leqslant n \leqslant \operatorname{len} g$ and g is a correct n-th step w.r.t. IPC (X), then $f \frown g$ is a correct $n + \operatorname{len} f$-th step w.r.t. IPC (X). The theorem is a consequence of (2).

(8) If f is a proof w.r.t. IPC (X) and g is a proof w.r.t. IPC (X), then $f \frown g$ is a proof w.r.t. IPC (X). The theorem is a consequence of (6) and (7).

(9) If f is a proof w.r.t. IPC (X) and $X \subseteq Y$, then f is a proof w.r.t. IPC (Y). The theorem is a consequence of (2).

(10) If f is a proof w.r.t. IPC (X) and $1 \leqslant l \leqslant \operatorname{len} f$, then $(f(l))_{\mathbf{1}} \in \operatorname{CnIPC}(X)$.

PROOF: For every n such that $1 \leqslant n \leqslant \operatorname{len} f$ holds $(f(n))_{\mathbf{1}} \in \operatorname{CnIPC}(X)$. □

Let us consider f. Assume $f \neq \emptyset$. The functor Effect-IPC(f) yielding an element of MC-w.f.f. is defined by the term

(Def. 5) $(f(\operatorname{len} f))_{\mathbf{1}}$.

Now we state the proposition:

(11) If f is a proof w.r.t. IPC (X), then Effect-IPC$(f) \in \operatorname{CnIPC}(X)$. The theorem is a consequence of (4) and (10).

2. A Consequence as a Set of All Intuitionistic Provable Formulas

Now we state the proposition:

(12) $X \subseteq \{F :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = F\}$. The theorem is a consequence of (1).

Let us consider X. Now we state the propositions:

(13) Suppose $Y = \{p :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = p\}$. Then Y is IPC theory.

(14) $\{p :$ there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC $(f) = p\} =$ CnIPC(X). The theorem is a consequence of (12) and (13).

(15) $p \in$ CnIPC(X) if and only if there exists f such that f is a proof w.r.t. IPC (X) and Effect-IPC$(f) = p$. The theorem is a consequence of (14).

(16) If $p \in$ CnIPC(X), then there exists Y such that $Y \subseteq X$ and Y is finite and $p \in$ CnIPC(Y).

PROOF: Consider f such that f is a proof w.r.t. IPC (X) and Effect-IPC(f) $= p$. Consider A being a set such that A is finite and $A \subseteq$ MC-w.f.f. and rng $f \subseteq A \times$ Proof-Step-Kinds-IPC. If $1 \leqslant n \leqslant$ len f, then f is a correct n-th step w.r.t. IPC (Y). □

3. The Intuitionistic Provable Relation

Let us consider X and s. We say that $X \vdash_{IPC} (s)$ if and only if

(Def. 6) $s \in$ CnIPC(X).

We say that $\vdash_{IPC} s$ if and only if

(Def. 7) $\emptyset_{\text{MC-w.f.f.}} \vdash_{IPC} s$.

Now we state the propositions:

(17) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow p))$.

(18) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r)))$.

(19) $X \vdash_{IPC} (p \wedge q \Rightarrow p)$.

(20) $X \vdash_{IPC} (p \wedge q \Rightarrow q)$.

(21) $X \vdash_{IPC} (p \Rightarrow (q \Rightarrow p \wedge q))$.

(22) $X \vdash_{IPC} (p \Rightarrow p \vee q)$.

(23) $X \vdash_{IPC} (q \Rightarrow p \vee q)$.

(24) $X \vdash_{IPC} (p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r)))$.

(25) $X \vdash_{IPC} (\text{FALSUM} \Rightarrow p)$.

(26) If $X \vdash_{IPC} p$ and $X \vdash_{IPC} (p \Rightarrow q)$, then $X \vdash_{IPC} (q)$.

(27) $\vdash_{IPC} p \Rightarrow (q \Rightarrow p)$.

(28) $\vdash_{IPC} p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$.

(29) $\vdash_{IPC} p \wedge q \Rightarrow p$.

(30) $\vdash_{IPC} p \wedge q \Rightarrow q$.

(31) $\vdash_{IPC} p \Rightarrow (q \Rightarrow p \wedge q)$.

(32) $\vdash_{IPC} p \Rightarrow p \vee q$.

(33) $\vdash_{IPC} q \Rightarrow p \vee q$.

(34) $\vdash_{IPC} p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$.

(35) $\vdash_{IPC}$ FALSUM $\Rightarrow p$.

(36) If $\vdash_{IPC} p$ and $\vdash_{IPC} p \Rightarrow q$, then $\vdash_{IPC} q$.

Let us consider s. We say that s is IPC-valid if and only if

(Def. 8) $\emptyset_{\text{MC-w.f.f.}} \vdash_{IPC}(s)$.

One can verify that s is IPC-valid if and only if the condition (Def. 9) is satisfied.

(Def. 9) $s \in$ IPC-Taut.

Now we state the propositions:

(37) If p is IPC-valid, then $X \vdash_{IPC}(p)$.

(38) $p \Rightarrow (q \Rightarrow p)$ is IPC-valid.

(39) $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$ is IPC-valid.

(40) $p \wedge q \Rightarrow p$ is IPC-valid.

(41) $p \wedge q \Rightarrow q$ is IPC-valid.

(42) $p \Rightarrow (q \Rightarrow p \wedge q)$ is IPC-valid.

(43) $p \Rightarrow p \vee q$ is IPC-valid.

(44) $q \Rightarrow p \vee q$ is IPC-valid.

(45) $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$ is IPC-valid.

(46) FALSUM $\Rightarrow p$ is IPC-valid.

(47) If p is IPC-valid and $p \Rightarrow q$ is IPC-valid, then q is IPC-valid.

In the sequel X, T denote subsets of MC-w.f.f., F, G, H, p, q, r, t denote elements of MC-w.f.f., s, h denote MC-formulas, f denotes a finite sequence of elements of MC-w.f.f. $\times$ Proof-Step-Kinds-IPC, and i, j denote elements of $\mathbb{N}$.

4. The First Deduction Theorem for IPC

Now we state the propositions:

(48) $X \vdash_{IPC}(p \Rightarrow p)$. The theorem is a consequence of (26).

(49) $X \vdash_{IPC}$(IVERUM).

(50) If $X \vdash_{IPC}(p)$, then $X \vdash_{IPC}(q \Rightarrow p)$.

(51) If p is IPC-valid, then $X \vdash_{IPC}(p)$.

(52) If $X \cup \{F\} \vdash_{IPC}(G)$, then $X \vdash_{IPC}(F \Rightarrow G)$.

Proof: Consider f such that f is a proof w.r.t. IPC $(X \cup \{F\})$ and Effect-IPC$(f) = G$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then for every H such that $H = (f(\$_1))_{\mathbf{1}}$ holds $X \vdash_{IPC}(F \Rightarrow H)$. For every natural number n such that for every natural number k such that $k < n$ holds $\mathcal{P}[k]$ holds $\mathcal{P}[n]$. For every natural number n, $\mathcal{P}[n]$. $1 \leqslant \operatorname{len} f$. □

5. A Family of Deduction Theorems for IPC

From now on F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, G denote MC-formulas and x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x denote elements of MC-w.f.f..

Let x_1, x_2, x_3 be elements of MC-w.f.f.. Let us observe that the functor $\{x_1, x_2, x_3\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4 be elements of MC-w.f.f.. One can check that the functor $\{x_1, x_2, x_3, x_4\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be elements of MC-w.f.f..

One can check that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 be elements of MC-w.f.f.. Let us note that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 be elements of MC-w.f.f.. One can verify that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ yields a subset of MC-w.f.f.. Let x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} be elements of MC-w.f.f.. Observe that the functor $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ yields a subset of MC-w.f.f.. Now we state the propositions:

(53) If $\{F\} \vdash_{IPC} (G)$, then $\vdash_{IPC} F \Rightarrow G$. The theorem is a consequence of (52).

(54) If $\{F_1, F_2\} \vdash_{IPC} (G)$, then $\{F_2\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(55) If $\{F_1, F_2, F_3\} \vdash_{IPC} (G)$, then $\{F_2, F_3\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(56) If $\{F_1, F_2, F_3, F_4\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(57) If $\{F_1, F_2, F_3, F_4, F_5\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4, F_5\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(58) If $\{F_1, F_2, F_3, F_4, F_5, F_6\} \vdash_{IPC} (G)$, then $\{F_2, F_3, F_4, F_5, F_6\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(59) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(60) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

(61) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (52).

From now on x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} denote objects.

Now we state the propositions:

(62) $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} \cup \{x_1\}$.

(63) Suppose $\{F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}\} \vdash_{IPC} (G)$. Then $\{F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}\} \vdash_{IPC} (F_1 \Rightarrow G)$. The theorem is a consequence of (62) and (52).

6. Intuitionistic Provable Formulas and Theorems

Now we state the propositions:

(64) $\{p\} \vdash_{IPC} (p)$.

(65) If $X \vdash_{IPC} (p)$ and $X \subseteq Y$, then $Y \vdash_{IPC} (p)$. The theorem is a consequence of (15) and (9).

(66) If $p \in X$, then $X \vdash_{IPC} (p)$. The theorem is a consequence of (64) and (65).

(67) If $p \in X$, then $p \in \mathrm{CnIPC}(X)$. The theorem is a consequence of (66).

(68) If $p \in$ IPC-Taut, then $\vdash_{IPC} p$.

(69) If $\vdash_{IPC} p$, then $p \in$ IPC-Taut.

(70) $p \in$ IPC-Taut if and only if $\vdash_{IPC} p$.

(71) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (26), (54), and (53).

(72) $\{p \wedge q\} \vdash_{IPC} (p)$. The theorem is a consequence of (19), (64), and (26).

(73) $\{p \wedge q\} \vdash_{IPC} (q)$. The theorem is a consequence of (20), (64), and (26).

(74) $\vdash_{IPC} (p \Rightarrow q) \wedge (p \Rightarrow (q \Rightarrow \mathrm{FALSUM})) \Rightarrow (p \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(75) $\vdash_{IPC} p \Rightarrow \mathrm{FALSUM} \Rightarrow (p \Rightarrow q)$. The theorem is a consequence of (68).

(76) $\vdash_{IPC} (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$. The theorem is a consequence of (72), (73), (24), (26), and (53).

(77) $\vdash_{IPC} p \wedge (p \Rightarrow q) \Rightarrow q$. The theorem is a consequence of (72), (73), (26), and (53).

(78) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM} \Rightarrow \mathrm{FALSUM})$. The theorem is a consequence of (69), (71), and (68).

(79) $\vdash_{IPC} (p \Rightarrow \mathrm{FALSUM}) \vee q \Rightarrow (p \Rightarrow q)$. The theorem is a consequence of (69), (75), (76), and (68).

(80) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}))$.

(81) $\vdash_{IPC} (p \Rightarrow \text{FALSUM}) \vee (q \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (69), (76), (80), and (68).

(82) Let us consider MC-formulas p, q. If $\vdash_{IPC} p$ and $\vdash_{IPC} q$, then $\vdash_{IPC} p \wedge q$. The theorem is a consequence of (31) and (36).

(83) If $\vdash_{IPC} p \Rightarrow q$ and $\vdash_{IPC} q \Rightarrow p$, then $\vdash_{IPC} p \Leftrightarrow q$.

(84) $\vdash_{IPC} p \Rightarrow p$. The theorem is a consequence of (27), (28), and (26).

(85) $\vdash_{IPC} p \Leftrightarrow p$. The theorem is a consequence of (84) and (82).

(86) $\vdash_{IPC} p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (55), (54), and (53).

(87) $\vdash_{IPC} p \Rightarrow (q \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(88) $\vdash_{IPC} (p \wedge q \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (86), (87), and (83).

(89) $\vdash_{IPC} p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow (p \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (55), (54), and (53).

(90) $\vdash_{IPC} q \Rightarrow (p \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(91) $\vdash_{IPC} (q \Rightarrow (p \Rightarrow \text{FALSUM})) \Leftrightarrow (p \wedge q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (89), (90), and (83).

(92) $\vdash_{IPC} p \Rightarrow (q \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(93) $\vdash_{IPC} q \Rightarrow (p \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(94) $\vdash_{IPC} p \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(95) $\vdash_{IPC} q \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (65), (26), (55), (54), and (53).

(96) $\vdash_{IPC} p \vee q \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (68).

(97) $\vdash_{IPC} (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM}) \Rightarrow (p \vee q \Rightarrow \text{FALSUM})$.

(98) $\vdash_{IPC} (p \vee q \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (96), (97), and (83).

(99) $\vdash_{IPC} p \wedge (p \Rightarrow \text{FALSUM}) \Rightarrow \text{FALSUM}$.

(100) $\vdash_{IPC} \text{FALSUM} \Leftrightarrow p \wedge (p \Rightarrow \text{FALSUM})$. The theorem is a consequence of (35), (99), and (83).

(101) $\vdash_{IPC} p \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$.

(102) $\vdash_{IPC} p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM})$. The theorem is a consequence of (69), (71), and (68).

(103) $\vdash_{IPC}(p \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$. The theorem is a consequence of (101), (102), and (83).

(104) $\vdash_{IPC} p \Rightarrow \text{FALSUM} \Rightarrow q \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow q)$. The theorem is a consequence of (66), (102), (65), (26), (54), and (53).

(105) $\vdash_{IPC} p \Rightarrow q \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (69), (80), and (68).

(106) $\vdash_{IPC} p \wedge (q \Rightarrow \text{FALSUM}) \Rightarrow (p \Rightarrow q \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (19), (26), (20), (54), and (53).

(107) $\vdash_{IPC} p \Rightarrow q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (21), (26), (106), (80), (36), (65), (56), (55), (54), and (53).

(108) $\vdash_{IPC} p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \Rightarrow (p \Rightarrow q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (79), (80), (36), (65), (26), (96), (19), (20), (54), and (53).

(109) $\vdash_{IPC}(p \Rightarrow q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (107), (108), and (83).

(110) $\vdash_{IPC} p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$. The theorem is a consequence of (29), (30), (80), (36), and (68).

(111) $\vdash_{IPC}(p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \Rightarrow (p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$. The theorem is a consequence of (66), (21), (26), (56), (19), (55), (20), (54), and (53).

(112) $\vdash_{IPC}(p \wedge q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \Leftrightarrow (p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}) \wedge (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM})$. The theorem is a consequence of (110), (111), and (83).

(113) $\vdash_{IPC} p \Rightarrow q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow (p \Rightarrow (q \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (107), (65), (26), (71), (54), and (53).

(114) If $\vdash_{IPC} r$ and $\{r\} \vdash_{IPC}(q)$, then $\vdash_{IPC} q$. The theorem is a consequence of (53) and (36).

(115) If $X \vdash_{IPC}(r)$ and $X \cup \{r\} \vdash_{IPC}(q)$, then $X \vdash_{IPC}(q)$. The theorem is a consequence of (52) and (26).

(116) If $X \vdash_{IPC}(r)$ and $Y \cup \{r\} \vdash_{IPC}(q)$, then $X \cup Y \vdash_{IPC}(q)$. The theorem is a consequence of (52), (65), and (26).

(117) If $\vdash_{IPC} p$ and $\{r\} \vdash_{IPC} (q)$, then $\{p \Rightarrow r\} \vdash_{IPC} (q)$. The theorem is a consequence of (65), (64), (26), and (115).

(118) If $X \vdash_{IPC} (p)$ and $X \cup \{r\} \vdash_{IPC} (q)$, then $X \cup \{p \Rightarrow r\} \vdash_{IPC} (q)$. The theorem is a consequence of (65), (66), (26), and (115).

(119) $\{q\} \vdash_{IPC} (q \vee r)$. The theorem is a consequence of (64), (22), and (26).

(120) $\{r\} \vdash_{IPC} (q \vee r)$. The theorem is a consequence of (64), (23), and (26).

(121) If $\{p\} \vdash_{IPC} (r)$ and $\{q\} \vdash_{IPC} (r)$, then $\{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (34), (53), (36), (65), (26), and (64).

(122) If $X \cup \{p\} \vdash_{IPC} (r)$ and $X \cup \{q\} \vdash_{IPC} (r)$, then $X \cup \{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (52), (24), (26), (64), and (65).

(123) If $X \cup \{p\} \vdash_{IPC} (r)$ and $Y \cup \{q\} \vdash_{IPC} (r)$, then $(X \cup Y) \cup \{p \vee q\} \vdash_{IPC} (r)$. The theorem is a consequence of (52), (65), (24), (26), and (64).

(124) $\vdash_{IPC} p \Rightarrow q \vee (p \Rightarrow r) \Rightarrow (p \Rightarrow q \vee r)$. The theorem is a consequence of (120), (65), (64), (118), (119), (122), (52), and (53).

(125) $\vdash_{IPC} p \Rightarrow (p \Rightarrow \text{FALSUM} \Rightarrow q)$. The theorem is a consequence of (66), (26), (25), (54), and (53).

(126) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \wedge r \Rightarrow \text{FALSUM} \Rightarrow (p \wedge r \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (20), (26), (19), (21), (55), (54), and (53).

(127) $\vdash_{IPC} p \Rightarrow q \Rightarrow (q \vee r \Rightarrow \text{FALSUM} \Rightarrow (p \vee r \Rightarrow \text{FALSUM}))$. The theorem is a consequence of (66), (68), (65), (26), (55), (54), and (53).

Let p be an element of MC-w.f.f.. Note that the functor $\text{neg}(p)$ yields an element of MC-w.f.f. and is defined by the term

(Def. 10) $p \Rightarrow \text{FALSUM}$.

The functor $\text{neg}^2(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 11) $p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}$.

The functor $\text{neg}^3(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 12) $p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}$.

The functor $\text{neg}^4(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 13) $p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}$.

The functor $\text{neg}^5(p)$ yielding an element of MC-w.f.f. is defined by the term

(Def. 14) $p \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM} \Rightarrow \text{FALSUM}$.

Now we state the propositions:

(128) $\vdash_{IPC} p \Rightarrow \text{neg}(\text{neg}(p))$.

(129) $\vdash_{IPC} p \Rightarrow \text{neg}^2(p)$.

(130) $\vdash_{IPC} (p \Rightarrow q) \wedge (p \Rightarrow \text{neg}(q)) \Rightarrow \text{neg}(p)$.

(131) $\vdash_{IPC} \text{neg}(p) \Rightarrow (p \Rightarrow q)$.

(132) $\vdash_{IPC} p \Rightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))))$.

(133) $\vdash_{IPC} \mathrm{neg}(p) \vee q \Rightarrow (p \Rightarrow q)$.

(134) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q) \Rightarrow \mathrm{neg}(p))$.

(135) $\vdash_{IPC} \mathrm{neg}(p) \vee \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \wedge q)$.

(136) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Rightarrow (p \Rightarrow \mathrm{neg}(q))$.

(137) $\vdash_{IPC} p \Rightarrow \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \wedge q)$.

(138) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Leftrightarrow (p \Rightarrow \mathrm{neg}(q))$.

(139) $\vdash_{IPC} \mathrm{neg}(p \wedge q) \Rightarrow (q \Rightarrow \mathrm{neg}(p))$.

(140) $\vdash_{IPC} q \Rightarrow \mathrm{neg}(p) \Rightarrow \mathrm{neg}(p \wedge q)$.

(141) $\vdash_{IPC} (q \Rightarrow \mathrm{neg}(p)) \Leftrightarrow \mathrm{neg}(p \wedge q)$.

(142) $\vdash_{IPC} p \Rightarrow (q \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q)))$.

(143) $\vdash_{IPC} q \Rightarrow (p \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q)))$.

(144) $\vdash_{IPC} p \Rightarrow (\mathrm{neg}(p \wedge q) \Rightarrow \mathrm{neg}(q))$.

(145) $\vdash_{IPC} q \Rightarrow (\mathrm{neg}(p \wedge q) \Rightarrow \mathrm{neg}(p))$.

(146) $\vdash_{IPC} \mathrm{neg}(p \vee q) \Rightarrow \mathrm{neg}(p) \wedge \mathrm{neg}(q)$.

(147) $\vdash_{IPC} \mathrm{neg}(p) \wedge \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \vee q)$.

(148) $\vdash_{IPC} \mathrm{neg}(p \vee q) \Leftrightarrow \mathrm{neg}(p) \wedge \mathrm{neg}(q)$.

(149) $\vdash_{IPC} p \wedge \mathrm{neg}(p) \Rightarrow \mathrm{FALSUM}$.

(150) $\vdash_{IPC} \mathrm{FALSUM} \Leftrightarrow p \wedge \mathrm{neg}(p)$.

(151) $\vdash_{IPC} \mathrm{neg}(p) \Rightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p)))$.

(152) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))) \Rightarrow \mathrm{neg}(p)$.

(153) $\vdash_{IPC} \mathrm{neg}(p) \Leftrightarrow \mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p)))$.

(154) $\vdash_{IPC} \mathrm{neg}(p) \Rightarrow q \Rightarrow (\mathrm{neg}(\mathrm{neg}(\mathrm{neg}(p))) \Rightarrow q)$.

(155) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(156) $\vdash_{IPC} p \wedge \mathrm{neg}(q) \Rightarrow \mathrm{neg}(p \Rightarrow q)$.

(157) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Rightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(158) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p \Rightarrow q))$.

(159) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Leftrightarrow (\mathrm{neg}(\mathrm{neg}(p)) \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(160) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \wedge q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q))$.

(161) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q)) \Rightarrow \mathrm{neg}(\mathrm{neg}(p \wedge q))$.

(162) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \wedge q)) \Leftrightarrow \mathrm{neg}(\mathrm{neg}(p)) \wedge \mathrm{neg}(\mathrm{neg}(q))$.

(163) $\vdash_{IPC} \mathrm{neg}(\mathrm{neg}(p \Rightarrow q)) \Rightarrow (p \Rightarrow \mathrm{neg}(\mathrm{neg}(q)))$.

(164) $\vdash_{IPC} p \Rightarrow (\mathrm{neg}(p) \Rightarrow q)$.

(165) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q \wedge r) \Rightarrow \mathrm{neg}(p \wedge r))$.

(166) $\vdash_{IPC} p \Rightarrow q \Rightarrow (\mathrm{neg}(q \vee r) \Rightarrow \mathrm{neg}(p \vee r))$.

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