FORMALIZED MATHEMATICS
Vol. 29, No. 4, Pages 241–248, 2021
DOI: 10.2478/forma-2021-0022

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The 3-Fold Product Space of Real Normed Spaces and its Properties

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Summary. In this article, we formalize in Mizar [1], [2] the 3-fold product space of real normed spaces for usefulness in application fields such as engineering, although the formalization of the 2-fold product space of real normed spaces has been stored in the Mizar Mathematical Library [3].

First, we prove some theorems about the 3-variable function and 3-fold Cartesian product for preparation. Then we formalize the definition of 3-fold product space of real linear spaces. Finally, we formulate the definition of 3-fold product space of real normed spaces. We referred to [7] and [6] in the formalization.

MSC: 46B15 46B20 68V20

Keywords: 3-fold product spaces; linear spaces; normed spaces

MML identifier: PRVECT_4, version: 8.1.11 5.68.1412

1. 3-Variable Function & 3-Fold Cartesian Product

From now on v, x, x_1, x_2, y, z denote objects and X, X_1, X_2, X_3 denote sets.

The scheme *FuncEx3A* deals with sets X, Y, W, Z and a 4-ary predicate P and states that

(Sch. 1) There exists a function f from $X \times Y \times W$ into Z such that for every objects x, y, w such that x, y, $w \in W$ holds $P[x, y, w, f(x, y, w)]$

provided

- for every objects x, y, w such that x, y, $w \in W$ there exists z such that $z \in Z$ and $P[x, y, w, z]$.

ISSN 1426–2630(Print), 1898-9934(Online)

Now we state the propositions:

(1) Let us consider non empty sets X, Y, Z, and a function D. Suppose $\mathrm{dom}\, D = \{1, 2, 3\}$ and $D(1) = X$ and $D(2) = Y$ and $D(3) = Z$. Then there exists a function I from $X \times Y \times Z$ into $\prod D$ such that

(i) I is one-to-one and onto, and

(ii) for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $I(x, y, z) = \langle x, y, z \rangle$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}, \text{object}, \text{object}] \equiv \$_4 = \langle \$_1, \$_2, \$_3 \rangle$. For every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ there exists an object w such that $w \in \prod D$ and $\mathcal{P}[x, y, z, w]$. Consider I being a function from $X \times Y \times Z$ into $\prod D$ such that for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $\mathcal{P}[x, y, z, I(x, y, z)]$. $\square$

(2) Let us consider non empty sets X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod \langle X, Y, Z \rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every objects x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ holds $I(x, y, z) = \langle x, y, z \rangle$.

The theorem is a consequence of (1).

2. 3-Fold Product Space of Real Linear Spaces

Let E, F, G be non empty additive loop structures. The functor $E \times F \times G$ yielding a strict, non empty additive loop structure is defined by the term

(Def. 1) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. One can verify that the functor $\langle e, f, g \rangle$ yields an element of $E \times F \times G$. Let E, F, G be Abelian, non empty additive loop structures. Observe that $E \times F \times G$ is Abelian.

Let E, F, G be add-associative, non empty additive loop structures. One can verify that $E \times F \times G$ is add-associative. Let E, F, G be right zeroed, non empty additive loop structures. Note that $E \times F \times G$ is right zeroed.

Let E, F, G be right complementable, non empty additive loop structures. Let us note that $E \times F \times G$ is right complementable.

Now we state the propositions:

(3) Let us consider non empty additive loop structures E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3 \rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$ by [5, (7)]. □

(4) Let us consider add-associative, right zeroed, right complementable, non empty additive loop structures E, F, G, a point x_1 of E, a point x_2 of F, and a point x_3 of G. Then $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$.

Let E, F, G be non empty RLS structures. The functor $E \times F \times G$ yielding a strict, non empty RLS structure is defined by the term

(Def. 2) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. Let us note that the functor $\langle e, f, g\rangle$ yields an element of $E \times F \times G$. Let E, F, G be Abelian, non empty RLS structures. One can check that $E \times F \times G$ is Abelian.

Let E, F, G be add-associative, non empty RLS structures. Let us note that $E \times F \times G$ is add-associative.

Let E, F, G be right zeroed, non empty RLS structures. Let us observe that $E \times F \times G$ is right zeroed. Let E, F, G be right complementable, non empty RLS structures. One can verify that $E \times F \times G$ is right complementable.

Now we state the propositions:

(5) Let us consider non empty RLS structures E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. □

(6) Let us consider add-associative, right zeroed, right complementable, non empty RLS structures E, F, G, a point x_1 of E, a point x_2 of F, and a point x_3 of G. Then $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$.

Let E, F, G be vector distributive, non empty RLS structures. Let us observe that $E \times F \times G$ is vector distributive.

Let E, F, G be scalar distributive, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar distributive.

Let E, F, G be scalar associative, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar associative.

Let E, F, G be scalar unital, non empty RLS structures. Let us observe that $E \times F \times G$ is scalar unital.

Let E, F, G be Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty RLS structures. One can verify that $\langle E, F, G\rangle$ is real-linear-space-yielding. Now we state the proposition:

(7) Let us consider real linear spaces X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every point x of X and for every point y of Y and for every point z of Z, $I(x, y, z) = \langle x, y, z\rangle$, and

(iii) for every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$, and

(iv) for every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$, and

(v) $I(0_{X \times Y \times Z}) = 0_{\prod\langle X,Y,Z\rangle}$.

PROOF: Set C_1 = the carrier of X. Set C_2 = the carrier of Y. Set C_3 = the carrier of Z. Consider I being a function from $C_1 \times C_2 \times C_3$ into $\prod\langle C_1, C_2, C_3\rangle$ such that I is one-to-one and onto and for every objects x, y, z such that $x \in C_1$ and $y \in C_2$ and $z \in C_3$ holds $I(x, y, z) = \langle x, y, z\rangle$. For every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$. For every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$. □

Let E, F, G be real linear spaces, e be a point of E, f be a point of F, and g be a point of G. Note that the functor $\langle e, f, g\rangle$ yields an element of $\prod\langle E, F, G\rangle$. Now we state the proposition:

(8) Let us consider real linear spaces E, F, G. Then

(i) for every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and

(v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$.

PROOF: Consider I being a function from $E \times F \times G$ into $\prod\langle E, F, G\rangle$ such that I is one-to-one and onto and for every point x of E and for every point y of F and for every point z of G, $I(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $E \times F \times G$, $I(v + w) = I(v) + I(w)$ and for every point v of $E \times F \times G$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle E,F,G\rangle} = I(0_{E\times F\times G})$.

For every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$. For every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$. $I(a \cdot \langle x_1, x_2, x_3\rangle) = I(a \cdot x_1, a \cdot x_2, a \cdot x_3)$. $\square$

3. 3-Fold Product Space of Real Normed Spaces

Let E, F, G be non empty normed structures. The functor $E\times F\times G$ yielding a strict, non empty normed structure is defined by the term

(Def. 3) $(E \times F) \times G$.

Let e be a point of E, f be a point of F, and g be a point of G. One can verify that the functor $\langle e, f, g\rangle$ yields an element of $E \times F \times G$. Let E, F, G be real normed spaces. Let us note that $E \times F \times G$ is reflexive, discernible, real normed space-like, scalar distributive, vector distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable and $\langle E, F, G\rangle$ is real-norm-space-yielding.

Now we state the propositions:

(9) Let us consider real normed spaces E, F, G. Then

(i) for every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and

(ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and

(iii) $0_{E\times F\times G} = \langle 0_E, 0_F, 0_G\rangle$, and

(iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$, and

(v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and

(vi) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2}$ and there exists an element w of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$.

PROOF: For every set x, x is a point of $E \times F \times G$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$. Consider v_{10} being an element of $\mathcal{R}^2$ such that $v_{10} = \langle \|\langle x_1, y_1\rangle\|, \|z1\|\rangle$ and $(\text{prodnorm}(E \times F, G))(\langle x_1, y_1\rangle, z1) = |v_{10}|$. Consider v_{20} being an element of $\mathcal{R}^2$ such that $v_{20} = \langle \|x_1\|, \|y_1\|\rangle$ and $(\text{prodnorm}(E, F))(x_1, y_1) = |v_{20}|$. □

(10) Let us consider real normed spaces X, Y, Z. Then there exists a function I from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$ such that

(i) I is one-to-one and onto, and

(ii) for every point x of X and for every point y of Y and for every point z of Z, $I(x, y, z) = \langle x, y, z\rangle$, and

(iii) for every points v, w of $X \times Y \times Z$, $I(v + w) = I(v) + I(w)$, and

(iv) for every point v of $X \times Y \times Z$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$, and

(v) $0_{\prod\langle X,Y,Z\rangle} = I(0_{X\times Y\times Z})$, and

(vi) for every point v of $X \times Y \times Z$, $\|I(v)\| = \|v\|$.

PROOF: Reconsider $X_0 = X$, $Y_0 = Y$, $Z_0 = Z$ as a real linear space. Consider I_0 being a function from $X_0 \times Y_0 \times Z_0$ into $\prod\langle X_0, Y_0, Z_0\rangle$ such that I_0 is one-to-one and onto and for every point x of X and for every point y of Y and for every point z of Z, $I_0(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $X_0 \times Y_0 \times Z_0$, $I_0(v + w) = I_0(v) + I_0(w)$ and for every point v of $X_0 \times Y_0 \times Z_0$ and for every real number r, $I_0(r \cdot v) = r \cdot I_0(v)$ and $0_{\prod\langle X_0,Y_0,Z_0\rangle} = I_0(0_{X_0 \times Y_0 \times Z_0})$.

Reconsider $I = I_0$ as a function from $X \times Y \times Z$ into $\prod\langle X, Y, Z\rangle$. For every points g_1, g_2 of $X_0 \times Y_0$ and for every points f_1, f_2 of Z_0, $(\text{prodadd}(X \times Y, Z))(\langle g_1, f_1\rangle, \langle g_2, f_2\rangle) = \langle g_1 + g_2, f_1 + f_2\rangle$. For every real number r and for every point g of $X_0 \times Y_0$ and for every point f of Z_0, $(\text{prodmlt}(X \times Y, Z))(r, \langle g, f\rangle) = \langle r \cdot g, r \cdot f\rangle$. For every point v of $X \times Y \times Z$, $\|I(v)\| = \|v\|$ by [4, (11)]. □

Let E, F, G be real normed spaces, e be a point of E, f be a point of F, and g be a point of G. One can check that the functor $\langle e, f, g\rangle$ yields an element of $\prod\langle E, F, G\rangle$. Now we state the proposition:

(11) Let us consider real normed spaces E, F, G. Then

- (i) for every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$, and
- (ii) for every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1, y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$, and
- (iii) $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$, and
- (iv) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $-\langle x_1, x_2, x_3\rangle = \langle -x_1, -x_2, -x_3\rangle$, and
- (v) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G and for every real number a, $a \cdot \langle x_1, x_2, x_3\rangle = \langle a \cdot x_1, a \cdot x_2, a \cdot x_3\rangle$, and
- (vi) for every point x_1 of E and for every point x_2 of F and for every point x_3 of G, $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2}$ and there exists an element w of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$.

PROOF: Consider I being a function from $E \times F \times G$ into $\prod\langle E, F, G\rangle$ such that I is one-to-one and onto and for every point x of E and for every point y of F and for every point z of G, $I(x, y, z) = \langle x, y, z\rangle$ and for every points v, w of $E \times F \times G$, $I(v + w) = I(v) + I(w)$ and for every point v of $E \times F \times G$ and for every real number r, $I(r \cdot v) = r \cdot I(v)$ and $0_{\prod\langle E,F,G\rangle} = I(0_{E \times F \times G})$ and for every point v of $E \times F \times G$, $\|I(v)\| = \|v\|$. For every set x, x is a point of $\prod\langle E, F, G\rangle$ iff there exists a point x_1 of E and there exists a point x_2 of F and there exists a point x_3 of G such that $x = \langle x_1, x_2, x_3\rangle$. For every points x_1, y_1 of E and for every points x_2, y_2 of F and for every points x_3, y_3 of G, $\langle x_1, x_2, x_3\rangle + \langle y_1,$

$y_2, y_3\rangle = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle$. $0_{\prod\langle E,F,G\rangle} = \langle 0_E, 0_F, 0_G\rangle$. $\|\langle x_1, x_2, x_3\rangle\| = \sqrt{\|x_1\|^{\mathbf{2}} + \|x_2\|^{\mathbf{2}} + \|x_3\|^{\mathbf{2}}}$. Consider w being an element of $\mathcal{R}^3$ such that $w = \langle \|x_1\|, \|x_2\|, \|x_3\|\rangle$ and $\|\langle x_1, x_2, x_3\rangle\| = |w|$. $\square$

Let E, F, G be complete real normed spaces. Let us note that $E \times F \times G$ is complete.

Acknowledgement: The authors would also like to express our gratitude to Prof. Yasunari Shidama for his support and encouragement.

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Accepted November 30, 2021