FORMALIZED MATHEMATICS
Vol. 29, No. 4, Pages 201–220, 2021
DOI: 10.2478/forma-2021-0019

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Improper Integral. Part I

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Summary. In this article, we deal with Riemann's improper integral [1], using the Mizar system [2], [3]. Improper integrals with finite values are discussed in [5] by Yamazaki et al., but in general, improper integrals do not assume that they are finite. Therefore, we have formalized general improper integrals that does not limit the integral value to a finite value. In addition, each theorem in [5] assumes that the domain of integrand includes a closed interval, but since the improper integral should be discusses based on the half-open interval, we also corrected it.

MSC: 26A42 68V20

Keywords: Improper integrals

MML identifier: INTEGR24, version: 8.1.11 5.68.1412

1. Preliminaries

Now we state the proposition:

(1) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $a \leqslant b \leqslant c$ and $[a,c] \subseteq \operatorname{dom} f$ and $f{\restriction}[a,b]$ is bounded and $f{\restriction}[b,c]$ is bounded and f is integrable on $[a,b]$ and f is integrable on $[b,c]$. Then

(i) f is integrable on $[a,c]$, and

(ii) $\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx$.

Let us consider a sequence s of real numbers. Now we state the propositions:

(2) If s is divergent to $+\infty$, then s is not divergent to $-\infty$ and s is not convergent.

(3) If s is divergent to $-\infty$, then s is not divergent to $+\infty$ and s is not convergent.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0. Now we state the propositions:

(4) Suppose f is left convergent in x_0 or left divergent to $+\infty$ in x_0 or left divergent to $-\infty$ in x_0. Then there exists a sequence s of real numbers such that

  (i) s is convergent, and

  (ii) $\lim s = x_0$, and

  (iii) $\operatorname{rng} s \subseteq \operatorname{dom} f \cap \,]-\infty, x_0[$.

  Proof: Define $\mathcal{F}[\text{natural number}, \text{real number}] \equiv x_0 - \frac{1}{\$_1+1} < \$_2 < x_0$ and $\$_2 \in \operatorname{dom} f$. For every element n of $\mathbb{N}$, there exists an element r of $\mathbb{R}$ such that $\mathcal{F}[n, r]$. Consider s being a sequence of real numbers such that for every element n of $\mathbb{N}$, $\mathcal{F}[n, s(n)]$. For every natural number n, $x_0 - \frac{1}{n+1} < s(n) < x_0$ and $s(n) \in \operatorname{dom} f$. □

(5) Suppose f is right convergent in x_0 or right divergent to $+\infty$ in x_0 or right divergent to $-\infty$ in x_0. Then there exists a sequence s of real numbers such that

  (i) s is convergent, and

  (ii) $\lim s = x_0$, and

  (iii) $\operatorname{rng} s \subseteq \operatorname{dom} f \cap \,]x_0, +\infty[$.

  Proof: Define $\mathcal{F}[\text{natural number}, \text{real number}] \equiv x_0 < \$_2 < x_0 + \frac{1}{\$_1+1}$ and $\$_2 \in \operatorname{dom} f$. For every element n of $\mathbb{N}$, there exists an element r of $\mathbb{R}$ such that $\mathcal{F}[n, r]$. Consider s being a sequence of real numbers such that for every element n of $\mathbb{N}$, $\mathcal{F}[n, s(n)]$. For every natural number n, $x_0 < s(n) < x_0 + \frac{1}{n+1}$ and $s(n) \in \operatorname{dom} f$. □

(6) If f is left divergent to $+\infty$ in x_0, then f is not left divergent to $-\infty$ in x_0 and f is not left convergent in x_0. The theorem is a consequence of (4) and (2).

(7) If f is left divergent to $-\infty$ in x_0, then f is not left divergent to $+\infty$ in x_0 and f is not left convergent in x_0. The theorem is a consequence of (4) and (3).

(8) If f is right divergent to $+\infty$ in x_0, then f is not right divergent to $-\infty$ in x_0 and f is not right convergent in x_0. The theorem is a consequence of (5) and (2).

(9) If f is right divergent to $-\infty$ in x_0, then f is not right divergent to $+\infty$ in x_0 and f is not right convergent in x_0. The theorem is a consequence of (5) and (3).

(10) Suppose f is right convergent in x_0. Then

(i) there exists a real number r such that $0 < r$ and $f\restriction]x_0, x_0 + r[$ is lower bounded, and

(ii) there exists a real number r such that $0 < r$ and $f\restriction]x_0, x_0 + r[$ is upper bounded.

Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0, x_0 + R[)$ holds $-1 + g < (f\restriction]x_0, x_0 + R[)(r_1)$. Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0, x_0 + R[)$ holds $(f\restriction]x_0, x_0 + R[)(r_1) < g + 1$. $\square$

(11) Suppose f is left convergent in x_0. Then

(i) there exists a real number r such that $0 < r$ and $f\restriction]x_0 - r, x_0[$ is lower bounded, and

(ii) there exists a real number r such that $0 < r$ and $f\restriction]x_0 - r, x_0[$ is upper bounded.

Proof: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0 - R, x_0[)$ holds $-1 + g < (f\restriction]x_0 - R, x_0[)(r_1)$. Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \operatorname{dom}(f\restriction]x_0 - R, x_0[)$ holds $(f\restriction]x_0 - R, x_0[)(r_1) < g + 1$. $\square$

(12) Suppose f is right divergent to $+\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f\restriction]x_0, x_0 + r[$ is lower bounded.

PROOF: Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $1 < f(r_1)$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \mathrm{dom}(f \restriction]x_0, x_0 + R[)$ holds $1 < (f \restriction]x_0, x_0 + R[)(r_1)$. □

(13) Suppose f is right divergent to $-\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f \restriction]x_0, x_0 + r[$ is upper bounded.

PROOF: Consider r being a real number such that $x_0 < r$ and for every real number r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $f(r_1) < 1$. Set $R = r - x_0$. For every object r_1 such that $r_1 \in \mathrm{dom}(f \restriction]x_0, x_0 + R[)$ holds $(f \restriction]x_0, x_0 + R[)(r_1) < 1$. □

(14) Suppose f is left divergent to $+\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f \restriction]x_0 - r, x_0[$ is lower bounded.

PROOF: Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \mathrm{dom}\, f$ holds $1 < f(r_1)$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \mathrm{dom}(f \restriction]x_0 - R, x_0[)$ holds $1 < (f \restriction]x_0 - R, x_0[)(r_1)$. □

(15) Suppose f is left divergent to $-\infty$ in x_0. Then there exists a real number r such that

(i) $0 < r$, and

(ii) $f \restriction]x_0 - r, x_0[$ is upper bounded.

PROOF: Consider r being a real number such that $r < x_0$ and for every real number r_1 such that $r < r_1 < x_0$ and $r_1 \in \mathrm{dom}\, f$ holds $f(r_1) < 1$. Set $R = x_0 - r$. For every object r_1 such that $r_1 \in \mathrm{dom}(f \restriction]x_0 - R, x_0[)$ holds $(f \restriction]x_0 - R, x_0[)(r_1) < 1$. □

Let us consider partial functions f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$ and a real number x_0.

(16) Suppose f_1 is right divergent to $-\infty$ in x_0 and for every real number r such that $x_0 < r$ there exists a real number g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 + f_2)$ and there exists a real number r such that $0 < r$ and $f_2 \restriction]x_0, x_0 + r[$ is upper bounded. Then $f_1 + f_2$ is right divergent to $-\infty$ in x_0.

(17) Suppose f_1 is left divergent to $-\infty$ in x_0 and for every real number r such that $r < x_0$ there exists a real number g such that $r < g < x_0$ and $g \in \mathrm{dom}(f_1 + f_2)$ and there exists a real number r such that $0 < r$ and

$f_2 \restriction]x_0 - r, x_0[$ is upper bounded. Then $f_1 + f_2$ is left divergent to $-\infty$ in x_0.

2. Properties of Extended Riemann Integral

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(18) Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then

(i) f is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

PROOF: Reconsider $A =]a, b]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}$(element of A) $= (\int\limits_{\$_1}^{b} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. Consider M_0 being a real number such that for every object x such that $x \in [a, b] \cap \operatorname{dom} f$ holds $|f(x)| \leqslant M_0$. Reconsider $M = M_0 + 1$ as a real number. For every real number x such that $x \in [a, b]$ holds $|f(x)| < M$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - \int\limits_{a}^{b} f(x)dx| < g_1$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom} I_1$. $\square$

(19) Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then

(i) f is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

PROOF: Reconsider $A = [a, b[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}$(element of A) $= (\int\limits_{a}^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. Consider M_0 being a real number such that for every object x such that $x \in [a, b] \cap \operatorname{dom} f$ holds $|f(x)| \leqslant M_0$. Reconsider $M = M_0 + 1$ as a real number. For every real number x such that $x \in [a, b]$ holds $|f(x)| < M$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < b$ and for every real number r_1 such that $r < r_1 < b$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - \int\limits_a^b f(x)dx| < g_1$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. For every real number r such that $r < b$ there exists a real number g such that $r < g < b$ and $g \in \operatorname{dom} I_1$. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(20) Suppose $a < b \leqslant c$ and $]a, c] \subseteq \operatorname{dom} f$ and $f{\restriction}[b, c]$ is bounded and f is integrable on $[b, c]$ and f is left extended Riemann integrable on a, b. Then

(i) f is left extended Riemann integrable on a, c, and

(ii) $(R^{<})\int\limits_a^c f(x)dx = (R^{<})\int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx$.

Proof: For every real number e such that $a < e \leqslant c$ holds f is integrable on $[e, c]$ and $f{\restriction}[e, c]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_x^b f(x)dx$ and I is right convergent in a. Reconsider $A =]a, c]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_{\$_1}^c f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^c f(x)dx$.

For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_b^c f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r

such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $a < r$ and for every real number r_1 such that $r_1 < r$ and $a < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - ((R^{<})\int\limits_a^b f(x)dx + \int\limits_b^c f(x)dx)| < g_1$. □

(21) Suppose $a \leqslant b < c$ and $[a, c[\subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$ and f is right extended Riemann integrable on b, c. Then

(i) f is right extended Riemann integrable on a, c, and

(ii) $(R^{>})\int\limits_a^c f(x)dx = \int\limits_a^b f(x)dx + (R^{>})\int\limits_b^c f(x)dx$.

PROOF: For every real number e such that $a \leqslant e < c$ holds f is integrable on $[a, e]$ and $f \restriction [a, e]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = [b, c[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_b^x f(x)dx$ and I is left convergent in c. Reconsider $A = [a, c[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_a^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. For every real number r such that $r < c$ there exists a real number g such that $r < g < c$ and $g \in \operatorname{dom} I_1$.

Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_a^b f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that $r < c$ and for every real number r_1 such that $r < r_1 < c$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - (\int\limits_a^b f(x)dx + (R^{>})\int\limits_b^c f(x)dx)| < g_1$. □

(22) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Let us consider a real number d. Suppose $a < d \leqslant b$. Then

(i) f is left extended Riemann integrable on a, d, and

(ii) $(R^{<})\int_{a}^{b} f(x)dx = (R^{<})\int_{a}^{d} f(x)dx + \int_{d}^{b} f(x)dx$.

The theorem is a consequence of (20).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and real numbers c, d. Now we state the propositions:

(23) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then suppose $a \leqslant c < d \leqslant b$. Then

(i) f is left extended Riemann integrable on c, d, and

(ii) if $a < c$, then $(R^{<})\int_{c}^{d} f(x)dx = \int_{c}^{d} f(x)dx$.

The theorem is a consequence of (22).

(24) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then if $a < c < d \leqslant b$, then f is right extended Riemann integrable on c, d and $(R^{>})\int_{c}^{d} f(x)dx = \int_{c}^{d} f(x)dx$. The theorem is a consequence of (19).

(25) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Let us consider a real number c. Suppose $a \leqslant c < b$. Then

(i) f is right extended Riemann integrable on c, b, and

(ii) $(R^{>})\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + (R^{>})\int_{c}^{b} f(x)dx$.

The theorem is a consequence of (21).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and real numbers c, d. Now we state the propositions:

(26) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then suppose $a \leqslant c < d \leqslant b$. Then

(i) f is right extended Riemann integrable on c, d, and

(ii) if $d < b$, then $(R^{>})\int_{c}^{d} f(x)dx = \int_{c}^{d} f(x)dx$.

The theorem is a consequence of (25).

(27) Suppose $[a,b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then if $a \leqslant c < d < b$, then f is left extended Riemann integrable on c, d and $(R^{<})\int\limits_{c}^{d} f(x)dx = \int\limits_{c}^{d} f(x)dx$. The theorem is a consequence of (18).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(28) Suppose $a < b$ and $]a,b] \subseteq \operatorname{dom} f$ and $]a,b] \subseteq \operatorname{dom} g$ and f is left extended Riemann integrable on a, b and g is left extended Riemann integrable on a, b. Then

(i) $f+g$ is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int\limits_{a}^{b}(f+g)(x)dx = (R^{<})\int\limits_{a}^{b} f(x)dx + (R^{<})\int\limits_{a}^{b} g(x)dx$.

Proof: Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_2 =]a,b]$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int\limits_{x}^{b} g(x)dx$ and I_2 is right convergent in a and $(R^{<})\int\limits_{a}^{b} g(x)dx = \lim_{a^{+}} I_2$. Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a,b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and I_1 is right convergent in a and $(R^{<})\int\limits_{a}^{b} f(x)dx = \lim_{a^{+}} I_1$. Set $I_3 = I_1 + I_2$. $\operatorname{dom} I_3 =]a,b]$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int\limits_{x}^{b}(f+g)(x)dx$. For every real number r such that $a < r$ there exists a real number g such that $g < r$ and $a < g$ and $g \in \operatorname{dom}(I_1 + I_2)$. For every real number d such that $a < d \leqslant b$ holds $f+g$ is integrable on $[d,b]$ and $(f+g){\restriction}[d,b]$ is bounded. □

(29) Suppose $a < b$ and $[a,b[\subseteq \operatorname{dom} f$ and $[a,b[\subseteq \operatorname{dom} g$ and f is right extended Riemann integrable on a, b and g is right extended Riemann integrable on a, b. Then

(i) $f+g$ is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int\limits_{a}^{b}(f+g)(x)dx = (R^{>})\int\limits_{a}^{b} f(x)dx + (R^{>})\int\limits_{a}^{b} g(x)dx$.

Proof: Consider I_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that

$\operatorname{dom} I_2 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_2$ holds $I_2(x) = \int_a^x g(x)dx$ and I_2 is left convergent in b and $(R^{>})\int_a^b g(x)dx = \lim_{b^-} I_2$. Consider I_1 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int_a^x f(x)dx$ and I_1 is left convergent in b and $(R^{>})\int_a^b f(x)dx = \lim_{b^-} I_1$. Set $I_3 = I_1 + I_2$. $\operatorname{dom} I_3 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_3$ holds $I_3(x) = \int_a^x (f+g)(x)dx$. For every real number r such that $r < b$ there exists a real number g such that $r < g < b$ and $g \in \operatorname{dom}(I_1 + I_2)$. For every real number d such that $a \leqslant d < b$ holds $f + g$ is integrable on $[a, d]$ and $(f + g)\restriction[a, d]$ is bounded. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a real number r. Now we state the propositions:

(30) Suppose $]a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Then

(i) $r \cdot f$ is left extended Riemann integrable on a, b, and

(ii) $(R^{<})\int_a^b (r \cdot f)(x)dx = r \cdot ((R^{<})\int_a^b f(x)dx)$.

Proof: For every real number r, $r \cdot f$ is left extended Riemann integrable on a, b and $(R^{<})\int_a^b (r \cdot f)(x)dx = r \cdot ((R^{<})\int_a^b f(x)dx)$. □

(31) Suppose $[a, b[\subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Then

(i) $r \cdot f$ is right extended Riemann integrable on a, b, and

(ii) $(R^{>})\int_a^b (r \cdot f)(x)dx = r \cdot ((R^{>})\int_a^b f(x)dx)$.

Proof: For every real number r, $r \cdot f$ is right extended Riemann integrable on a, b and $(R^{>})\int_a^b (r \cdot f)(x)dx = r \cdot ((R^{>})\int_a^b f(x)dx)$. □

3. Definition of Improper Integral

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. We say that f is left improper integrable on a and b if and only if

(Def. 1) for every real number d such that $a < d \leqslant b$ holds f is integrable on $[d, b]$ and $f{\restriction}[d, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and (I_1 is right convergent in a or right divergent to $+\infty$ in a or I_1 is right divergent to $-\infty$ in a).

We say that f is right improper integrable on a and b if and only if

(Def. 2) for every real number d such that $a \leqslant d < b$ holds f is integrable on $[a, d]$ and $f{\restriction}[a, d]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and (I_1 is left convergent in b or left divergent to $+\infty$ in b or I_1 is left divergent to $-\infty$ in b).

Assume f is left improper integrable on a and b. The functor left-improper-integral(f, a, b) yielding an extended real is defined by

(Def. 3) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and (I_1 is right convergent in a and $it = \lim_{a^+} I_1$ or I_1 is right divergent to $+\infty$ in a and $it = +\infty$ or I_1 is right divergent to $-\infty$ in a and $it = -\infty$).

Assume f is right improper integrable on a and b. The functor right-improper-integral(f, a, b) yielding an extended real is defined by

(Def. 4) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and (I_1 is left convergent in b and $it = \lim_{b^-} I_1$ or I_1 is left divergent to $+\infty$ in b and $it = +\infty$ or I_1 is left divergent to $-\infty$ in b and $it = -\infty$).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(32) If f is left extended Riemann integrable on a, b, then f is left improper integrable on a and b.

(33) If f is right extended Riemann integrable on a, b, then f is right improper integrable on a and b.

(34) Suppose f is left improper integrable on a and b. Then

(i) f is left extended Riemann integrable on a, b and left-improper-integral $(f, a, b) = (R^{<})\int\limits_{a}^{b} f(x)dx$, or

(ii) f is not left extended Riemann integrable on a, b and left-improper-integral$(f, a, b) = +\infty$, or

(iii) f is not left extended Riemann integrable on a, b and left-improper-integral$(f, a, b) = -\infty$.

The theorem is a consequence of (8) and (9).

(35) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and I_1 is right divergent to $+\infty$ in a or right divergent to $-\infty$ in a. Then f is not left extended Riemann integrable on a, b. The theorem is a consequence of (8) and (9).

(36) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is left improper integrable on a and b and $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$ and I_1 is right convergent in a. Then left-improper-integral$(f, a, b) = \lim_{a^+} I_1$. The theorem is a consequence of (34).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(37) Suppose $a < b \leqslant c$ and $]a, c] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and c. Then

(i) f is left improper integrable on a and b, and

(ii) if left-improper-integral$(f, a, c) = (R^{<})\int\limits_{a}^{c} f(x)dx$, then left-improper-integral$(f, a, b) = (R^{<})\int\limits_{a}^{b} f(x)dx$, and

(iii) if left-improper-integral$(f, a, c) = +\infty$, then left-improper-integral $(f, a, b) = +\infty$, and

(iv) if left-improper-integral$(f, a, c) = -\infty$, then left-improper-integral $(f, a, b) = -\infty$.

The theorem is a consequence of (34).

(38) Suppose $a < b \leqslant c$ and $]a, c] \subseteq \operatorname{dom} f$ and $f\restriction[b, c]$ is bounded and f is left improper integrable on a and b and f is integrable on $[b, c]$. Then

- (i) f is left improper integrable on a and c, and
- (ii) if left-improper-integral$(f, a, b) = (R^{<})\int\limits_{a}^{b} f(x)dx$, then left-improper-integral$(f, a, c) =$ left-improper-integral$(f, a, b) + \int\limits_{b}^{c} f(x)dx$, and
- (iii) if left-improper-integral$(f, a, b) = +\infty$, then left-improper-integral $(f, a, c) = +\infty$, and
- (iv) if left-improper-integral$(f, a, b) = -\infty$, then left-improper-integral $(f, a, c) = -\infty$.

The theorem is a consequence of (34).

(39) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is right improper integrable on a and b. Then

- (i) f is right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = (R^{>})\int\limits_{a}^{b} f(x)dx$, or
- (ii) f is not right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = +\infty$, or
- (iii) f is not right extended Riemann integrable on a, b and right-improper-integral$(f, a, b) = -\infty$.

The theorem is a consequence of (6) and (7).

(40) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$ and I_1 is left divergent to $+\infty$ in b or left divergent to $-\infty$ in b. Then f is not right extended Riemann integrable on a, b. The theorem is a consequence of (6) and (7).

(41) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b. Suppose f is right improper integrable on a and b and $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$ and

I_1 is left convergent in b. Then right-improper-integral$(f, a, b) = \lim_{b^-} I_1$. The theorem is a consequence of (39).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, c.

(42) Suppose $a \leqslant b < c$ and $[a, c[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and c. Then

(i) f is right improper integrable on b and c, and

(ii) if right-improper-integral$(f, a, c) = (R^{>})\int\limits_{a}^{c} f(x)dx$, then right-improper-integral$(f, b, c) = (R^{>})\int\limits_{b}^{c} f(x)dx$, and

(iii) if right-improper-integral$(f, a, c) = +\infty$, then right-improper-integral$(f, b, c) = +\infty$, and

(iv) if right-improper-integral$(f, a, c) = -\infty$, then right-improper-integral$(f, b, c) = -\infty$.

The theorem is a consequence of (39).

(43) Suppose $a \leqslant b < c$ and $[a, c[\subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is right improper integrable on b and c and f is integrable on $[a, b]$. Then

(i) f is right improper integrable on a and c, and

(ii) if right-improper-integral$(f, b, c) = (R^{>})\int\limits_{b}^{c} f(x)dx$, then right-improper-integral$(f, a, c) =$ right-improper-integral$(f, b, c) + \int\limits_{a}^{b} f(x)dx$, and

(iii) if right-improper-integral$(f, b, c) = +\infty$, then right-improper-integral$(f, a, c) = +\infty$, and

(iv) if right-improper-integral$(f, b, c) = -\infty$, then right-improper-integral$(f, a, c) = -\infty$.

The theorem is a consequence of (39).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, c be real numbers. We say that f is improper integrable on a and c if and only if

(Def. 5) there exists a real number b such that $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$.

Now we state the propositions:

(44) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose f is improper integrable on a and c. Then there exists a real number b such that

(i) $a < b < c$, and

(ii) left-improper-integral$(f, a, b) = (R^{<})\int_a^b f(x)dx$ and right-improper-integral$(f, b, c) = (R^{>})\int_b^c f(x)dx$ or left-improper-integral(f, a, b) + right-improper-integral$(f, b, c) = +\infty$ or left-improper-integral (f, a, b) + right-improper-integral$(f, b, c) = -\infty$.

The theorem is a consequence of (34) and (39).

(45) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $]a, c[\ \subseteq \operatorname{dom} f$ and $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$. Let us consider a real number b_1. Suppose $a < b_1 \leqslant b$. Then left-improper-integral(f, a, b) + right-improper-integral(f, b, c) = left-improper-integral(f, a, b_1) + right-improper-integral (f, b_1, c). The theorem is a consequence of (34) and (39).

(46) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, b, c. Suppose $]a, c[\ \subseteq \operatorname{dom} f$ and $a < b < c$ and f is left improper integrable on a and b and f is right improper integrable on b and c and it is not true that left-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(f, b, c) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(f, b, c) = -\infty$. Let us consider a real number b_2. Suppose $b \leqslant b_2 < c$. Then left-improper-integral(f, a, b) + right-improper-integral(f, b, c) = left-improper-integral(f, a, b_2) + right-improper-integral (f, b_2, c). The theorem is a consequence of (39) and (34).

(47) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose $]a, c[\ \subseteq \operatorname{dom} f$ and f is improper integrable on a and c. Let us consider real numbers b_1, b_2. Suppose $a < b_1 < c$ and $a < b_2 < c$. Then left-improper-integral(f, a, b_1) + right-improper-integral(f, b_1, c) = left-improper-integral(f, a, b_2) + right-improper-integral(f, b_2, c). The theorem is a consequence of (45) and (46).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Assume $]a, b[\subseteq \operatorname{dom} f$ and f is improper integrable on a and b. The functor improper-integral(f, a, b) yielding an extended real is defined by

(Def. 6) there exists a real number c such that $a < c < b$ and f is left improper integrable on a and c and f is right improper integrable on c and b and $\mathit{it} =$ left-improper-integral(f, a, c) + right-improper-integral(f, c, b).

Now we state the proposition:

(48) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, c. Suppose $]a, c[\subseteq \operatorname{dom} f$ and f is improper integrable on a and c. Let us consider a real number b. Suppose $a < b < c$. Then

(i) f is left improper integrable on a and b, and

(ii) f is right improper integrable on b and c, and

(iii) improper-integral$(f, a, c) =$ left-improper-integral(f, a, b) + right-improper-integral(f, b, c).

The theorem is a consequence of (37), (43), (47), (38), and (42).

4. Linearity of Improper Integral

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, real numbers a, b, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(49) Suppose f is left improper integrable on a and b and left-improper-integral $(f, a, b) = +\infty$. Then suppose $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. Then I_1 is right divergent to $+\infty$ in a.

(50) Suppose f is left improper integrable on a and b and left-improper-integral $(f, a, b) = -\infty$. Then suppose $\operatorname{dom} I_1 =]a, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. Then I_1 is right divergent to $-\infty$ in a.

(51) Suppose f is right improper integrable on a and b and right-improper-integral$(f, a, b) = +\infty$. Then suppose $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. Then I_1 is left divergent to $+\infty$ in b.

(52) Suppose f is right improper integrable on a and b and right-improper-integral$(f, a, b) = -\infty$. Then suppose $\operatorname{dom} I_1 = [a, b[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$. Then I_1 is left divergent to $-\infty$ in b.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, r.

(53) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b. Then

(i) $r \cdot f$ is left improper integrable on a and b, and

(ii) left-improper-integral$(r \cdot f, a, b) = r \cdot$ left-improper-integral(f, a, b).

PROOF: For every real number d such that $a < d \leqslant b$ holds $r \cdot f$ is integrable on $[d, b]$ and $(r \cdot f)\restriction[d, b]$ is bounded. □

(54) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b. Then

(i) $r \cdot f$ is right improper integrable on a and b, and

(ii) right-improper-integral$(r \cdot f, a, b) = r \cdot$right-improper-integral$(f, a, b)$.

PROOF: For every real number d such that $a \leqslant d < b$ holds $r \cdot f$ is integrable on $[a, d]$ and $(r \cdot f)\restriction[a, d]$ is bounded. □

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(55) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and f is left improper integrable on a and b. Then

(i) $-f$ is left improper integrable on a and b, and

(ii) left-improper-integral$(-f, a, b) = -$left-improper-integral(f, a, b).

The theorem is a consequence of (53).

(56) Suppose $a < b$ and $[a, b[\subseteq \operatorname{dom} f$ and f is right improper integrable on a and b. Then

(i) $-f$ is right improper integrable on a and b, and

(ii) right-improper-integral$(-f, a, b) = -$right-improper-integral(f, a, b).

The theorem is a consequence of (54).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(57) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and $]a, b] \subseteq \operatorname{dom} g$ and f is left improper integrable on a and b and g is left improper integrable on a and b and it is not true that left-improper-integral$(f, a, b) = +\infty$ and left-improper-integral$(g, a, b) = -\infty$ and it is not true that left-improper-integral$(f, a, b) = -\infty$ and left-improper-integral$(g, a, b) = +\infty$. Then

(i) $f + g$ is left improper integrable on a and b, and

(ii) left-improper-integral$(f + g, a, b)$ = left-improper-integral(f, a, b) + left-improper-integral(g, a, b).

PROOF: For every real number d such that $a < d \leqslant b$ holds $f + g$ is integrable on $[d, b]$ and $(f + g){\restriction}[d, b]$ is bounded. □

(58) Suppose $a < b$ and $[a, b[\ \subseteq \operatorname{dom} f$ and $[a, b[\ \subseteq \operatorname{dom} g$ and f is right improper integrable on a and b and g is right improper integrable on a and b and it is not true that right-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(g, a, b) = -\infty$ and it is not true that right-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(g, a, b) = +\infty$. Then

(i) $f + g$ is right improper integrable on a and b, and

(ii) right-improper-integral$(f+g, a, b)$ = right-improper-integral(f, a, b)+ right-improper-integral(g, a, b).

PROOF: For every real number d such that $a \leqslant d < b$ holds $f + g$ is integrable on $[a, d]$ and $(f + g){\restriction}[a, d]$ is bounded by [4, (11)]. □

(59) Suppose $a < b$ and $]a, b] \subseteq \operatorname{dom} f$ and $]a, b] \subseteq \operatorname{dom} g$ and f is left improper integrable on a and b and g is left improper integrable on a and b and it is not true that left-improper-integral$(f, a, b) = +\infty$ and left-improper-integral$(g, a, b) = +\infty$ and it is not true that left-improper-integral$(f, a, b) = -\infty$ and left-improper-integral$(g, a, b) = -\infty$. Then

(i) $f - g$ is left improper integrable on a and b, and

(ii) left-improper-integral$(f - g, a, b)$ = left-improper-integral(f, a, b) − left-improper-integral(g, a, b).

The theorem is a consequence of (55) and (57).

(60) Suppose $a < b$ and $[a, b[\ \subseteq \operatorname{dom} f$ and $[a, b[\ \subseteq \operatorname{dom} g$ and f is right improper integrable on a and b and g is right improper integrable on a and b and it is not true that right-improper-integral$(f, a, b) = +\infty$ and right-improper-integral$(g, a, b) = +\infty$ and it is not true that right-improper-integral$(f, a, b) = -\infty$ and right-improper-integral$(g, a, b) = -\infty$. Then

(i) $f - g$ is right improper integrable on a and b, and

(ii) right-improper-integral$(f-g, a, b)$ = right-improper-integral(f, a, b)− right-improper-integral(g, a, b).

The theorem is a consequence of (56) and (58).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b, r.

(61) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and f is improper integrable on a and b. Then

(i) $r \cdot f$ is improper integrable on a and b, and

(ii) improper-integral$(r \cdot f, a, b) = r \cdot$ improper-integral(f, a, b).

The theorem is a consequence of (48), (53), and (54).

(62) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and f is improper integrable on a and b. Then

(i) $-f$ is improper integrable on a and b, and

(ii) improper-integral$(-f, a, b) = -$improper-integral(f, a, b).

The theorem is a consequence of (61).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b.

(63) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and $]a, b[\subseteq \mathrm{dom}\, g$ and f is improper integrable on a and b and g is improper integrable on a and b and it is not true that improper-integral$(f, a, b) = +\infty$ and improper-integral$(g, a, b) = -\infty$ and it is not true that improper-integral$(f, a, b) = -\infty$ and improper-integral$(g, a, b) = +\infty$. Then

(i) $f + g$ is improper integrable on a and b, and

(ii) improper-integral$(f + g, a, b) =$ improper-integral$(f, a, b) +$ improper-integral(g, a, b).

The theorem is a consequence of (37), (38), (43), (42), (48), (57), and (58).

(64) Suppose $]a, b[\subseteq \mathrm{dom}\, f$ and $]a, b[\subseteq \mathrm{dom}\, g$ and f is improper integrable on a and b and g is improper integrable on a and b and it is not true that improper-integral$(f, a, b) = +\infty$ and improper-integral$(g, a, b) = +\infty$ and it is not true that improper-integral$(f, a, b) = -\infty$ and improper-integral$(g, a, b) = -\infty$. Then

(i) $f - g$ is improper integrable on a and b, and

(ii) improper-integral$(f - g, a, b) =$ improper-integral$(f, a, b) -$ improper-integral(g, a, b).

The theorem is a consequence of (62) and (63).

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Accepted September 30, 2021