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Relationship between the Riemann and Lebesgue Integrals

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Summary. The goal of this article is to clarify the relationship between Riemann and Lebesgue integrals. In previous article [5], we constructed a one-dimensional Lebesgue measure. The one-dimensional Lebesgue measure provides a measure of any intervals, which can be used to prove the well-known relationship [6] between the Riemann and Lebesgue integrals [1]. We also proved the relationship between the integral of a given measure and that of its complete measure. As the result of this work, the Lebesgue integral of a bounded real valued function in the Mizar system [2], [3] can be calculated by the Riemann integral.

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1. Preliminaries

Let us consider a non empty set X and a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(1) (i) $\operatorname{rng}\max_+(f) \subseteq \operatorname{rng} f \cup \{0\}$, and

(ii) $\operatorname{rng}\max_-(f) \subseteq \operatorname{rng}(-f) \cup \{0\}$.

(2) If f is real-valued, then $-f$ is real-valued and $\max_+(f)$ is real-valued and $\max_-(f)$ is real-valued. The theorem is a consequence of (1).

(3) If f is without $-\infty$ and without $+\infty$, then f is a partial function from X to $\mathbb{R}$.

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(4) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then

(i) $\max_+(f)$ is simple function in S, and

(ii) $\max_-(f)$ is simple function in S.

PROOF: Consider F being a finite sequence of separated subsets of S such that $\operatorname{dom} f = \bigcup \operatorname{rng} F$ and for every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $(\max_+(f))(x) = (\max_+(f))(y)$. For every natural number n and for every elements x, y of X such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $(\max_-(f))(x) = (\max_-(f))(y)$. □

Let us consider real numbers a, b. Now we state the propositions:

(5) Suppose $a \leqslant b$. Then

(i) (B-Meas)($[a, b]$) $= b - a$, and

(ii) (B-Meas)($[a, b[$) $= b - a$, and

(iii) (B-Meas)($]a, b]$) $= b - a$, and

(iv) (B-Meas)($]a, b[$) $= b - a$, and

(v) (L-Meas)($[a, b]$) $= b - a$, and

(vi) (L-Meas)($[a, b[$) $= b - a$, and

(vii) (L-Meas)($]a, b]$) $= b - a$, and

(viii) (L-Meas)($]a, b[$) $= b - a$.

(6) Suppose $a > b$. Then

(i) (B-Meas)($[a, b]$) $= 0$, and

(ii) (B-Meas)($[a, b[$) $= 0$, and

(iii) (B-Meas)($]a, b]$) $= 0$, and

(iv) (B-Meas)($]a, b[$) $= 0$, and

(v) (L-Meas)($[a, b]$) $= 0$, and

(vi) (L-Meas)($[a, b[$) $= 0$, and

(vii) (L-Meas)($]a, b]$) $= 0$, and

(viii) (L-Meas)($]a, b[$) $= 0$.

(7) Let us consider an element A_1 of the Borel sets, an element A_2 of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $A_1 = A_2$ and f is A_1-measurable, then f is A_2-measurable.

(8) Let us consider real numbers a, b, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $a < b$ and $A = [a, b]$. Let us consider a natural number n. If $n > 0$, then there exists a partition D of A such that D divides into equal n.

Let F be a finite sequence of elements of the Borel sets and n be a natural number. One can check that the functor $F(n)$ yields an extended real-membered set. Now we state the proposition:

(9) Let us consider real numbers a, b, a non empty, closed interval subset A of $\mathbb{R}$, and a partition D of A. Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets such that

(i) $\operatorname{dom} D = \operatorname{dom} F$, and

(ii) $\bigcup \operatorname{rng} F = A$, and

(iii) for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

Proof: Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ if $\operatorname{len} D = 1$, then $\$_2 = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $\$_1 = 1$, then $\$_2 = [a, D(\$_1)[$ and if $1 < \$_1 < \operatorname{len} D$, then $\$_2 = [D(\$_1 -' 1), D(\$_1)[$ and if $\$_1 = \operatorname{len} D$, then $\$_2 = [D(\$_1 -' 1), D(\$_1)]$. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} D$ there exists an element x of the Borel sets such that $\mathcal{P}[k, x]$ by [4, (5)]. Consider F being a finite sequence of elements of the Borel sets such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} D$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} D$ holds $\mathcal{P}[k, F(k)]$. For every objects x, y such that $x \neq y$ holds $F(x)$ misses $F(y)$. For every natural number k such that $k \in \operatorname{dom} F$ and $k \neq \operatorname{len} D$ holds $\bigcup \operatorname{rng}(F \restriction k) = [a, D(k)[$. $\bigcup \operatorname{rng} F = A$. $\square$

Let us consider real numbers a, b, a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, and a partial function f from A to $\mathbb{R}$. Now we state the propositions:

(10) Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$.

Proof: Consider F being a finite sequence of separated subsets of the Borel sets such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $\$_1 \in F(k)$ and $\$_2 = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Consider g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that for every object x, $x \in \operatorname{dom} g$ iff $x \in \mathbb{R}$ and there exists an object y such that $\mathcal{P}[x, y]$ and for every object x such that $x \in \operatorname{dom} g$ holds $\mathcal{P}[x, g(x)]$. For every natural number k and for every elements x, y of $\mathbb{R}$ such that $k \in \operatorname{dom} F$ and $x, y \in F(k)$ holds $g(x) = g(y)$. $\square$

(11) Suppose $A = [a, b]$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$.

Proof: Consider F being a finite sequence of separated subsets of the Borel sets such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$.

Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $\$_1 \in F(k)$ and $\$_2 = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Consider g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that for every object x, $x \in \operatorname{dom} g$ iff $x \in \mathbb{R}$ and there exists an object y such that $\mathcal{P}[x, y]$ and for every object x such that $x \in \operatorname{dom} g$ holds $\mathcal{P}[x, g(x)]$. For every natural number k and for every elements x, y of $\mathbb{R}$ such that $k \in \operatorname{dom} F$ and $x, y \in F(k)$ holds $g(x) = g(y)$. $\square$

(12) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, a finite sequence F of separated subsets of S, a finite sequence a of elements of $\overline{\mathbb{R}}$, and a natural number n. Suppose f is simple function in S and F and a are representation of f and $n \in \operatorname{dom} F$. Then

(i) $F(n) = \emptyset$, or

(ii) $a(n)$ is a real number.

Let A be a non empty, closed interval subset of $\mathbb{R}$ and n be a natural number. Assume $n > 0$ and $\mathrm{vol}(A) > 0$. The functor $\mathrm{EqDiv}(A, n)$ yielding a partition of A is defined by

(Def. 1) *it* divides into equal n.

Now we state the propositions:

(13) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a natural number n. If $\mathrm{vol}(A) > 0$ and $\mathrm{len}\,\mathrm{EqDiv}(A, 2^n) = 1$, then $n = 0$.

(14) Let us consider real numbers a, b, and a non empty, closed interval subset A of $\mathbb{R}$. Suppose $a < b$ and $A = [a, b]$. Then there exists a division sequence D of A such that for every natural number n, $D(n)$ divides into equal 2^n.
Proof: Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ there exists a partition D of A such that $D = \$_2$ and D divides into equal $2^{\$_1}$. For every element n of $\mathbb{N}$, there exists an element D of $\mathrm{divs}\,A$ such that $\mathcal{P}[n, D]$. Consider D being a function from $\mathbb{N}$ into $\mathrm{divs}\,A$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, D(n)]$. For every natural number n, $D(n)$ divides into equal 2^n. □

(15) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, and natural numbers n, k. Suppose D divides into equal n and $k \in \mathrm{dom}\,D$. Then $\mathrm{vol}(\mathrm{divset}(D, k)) = \frac{\mathrm{vol}(A)}{n}$.

(16) Let us consider a complex number x, and a natural number r. If $x \neq 0$, then $(x^r)^{-1} = (x^{-1})^r$.

(17) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a sequence T of $\mathrm{divs}\,A$. Suppose $\mathrm{vol}(A) > 0$ and for every natural number n, $T(n) = \mathrm{EqDiv}(A, 2^n)$. Then δ_T is 0-convergent and non-zero.
Proof: For every natural number n, $(\delta_T)(n) = 2 \cdot (\mathrm{vol}(A)) \cdot ((2^{-1})^{n+1})$. Define $\mathcal{S}(\text{natural number}) = (2^{-1})^{\$_1+1}$. Consider s being a sequence of real numbers such that for every natural number n, $s(n) = \mathcal{S}(n)$. For every natural number n, $(\delta_T)(n) = 2 \cdot (\mathrm{vol}(A)) \cdot s(n)$. □

(18) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, a partial function f from X to $\overline{\mathbb{R}}$, a finite sequence F of separated subsets of S, and finite sequences a, x of elements of $\overline{\mathbb{R}}$. Suppose f is simple function in S and $E = \mathrm{dom}\,f$ and $M(E) < +\infty$ and F and a are representation of f and $\mathrm{dom}\,x = \mathrm{dom}\,F$ and for every natural number i such that $i \in \mathrm{dom}\,x$ holds $x(i) = a(i) \cdot (M \cdot F)(i)$. Then $\int f\,\mathrm{d}M = \sum x$.
Proof: $\max_+(f)$ is simple function in S and $\max_-(f)$ is simple function in S. Define $\mathcal{P}[\text{natural number}, \text{extended real}] \equiv$ for every object x such that $x \in F(\$_1)$ holds $\$_2 = \max(f(x), 0)$. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\,a$ there exists an element y of $\overline{\mathbb{R}}$ such that $\mathcal{P}[k, y]$. Consider

a_1 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, a_1 = \mathrm{Seg}\,\mathrm{len}\, a$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, a$ holds $\mathcal{P}[k, a_1(k)]$. For every natural number k such that $k \in \mathrm{dom}\, F$ for every object x such that $x \in F(k)$ holds $(\max_+(f))(x) = a_1(k)$. Define $\mathcal{Q}[$natural number, extended real$] \equiv \$_2 = a_1(\$_1) \cdot (M \cdot F)(\$_1)$. Consider x_1 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, x_1 = \mathrm{Seg}\,\mathrm{len}\, F$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $\mathcal{Q}[k, x_1(k)]$. Reconsider $r_1 = x_1$ as a finite sequence of elements of $\mathbb{R}$. $\int' \max_+(f)\, \mathrm{d}M = \sum x_1$.

Define $\mathcal{P}[$natural number, extended real$] \equiv$ for every object x such that $x \in F(\$_1)$ holds $\$_2 = \max(-f(x), 0)$. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, a$ there exists an element y of $\overline{\mathbb{R}}$ such that $\mathcal{P}[k, y]$. Consider a_2 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, a_2 = \mathrm{Seg}\,\mathrm{len}\, a$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, a$ holds $\mathcal{P}[k, a_2(k)]$. For every natural number k such that $k \in \mathrm{dom}\, F$ for every object x such that $x \in F(k)$ holds $(\max_-(f))(x) = a_2(k)$. Define $\mathcal{Q}[$natural number, extended real$] \equiv \$_2 = a_2(\$_1) \cdot (M \cdot F)(\$_1)$. Consider x_2 being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, x_2 = \mathrm{Seg}\,\mathrm{len}\, F$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $\mathcal{Q}[k, x_2(k)]$. Reconsider $r_2 = x_2$ as a finite sequence of elements of $\mathbb{R}$. $\int' \max_-(f)\, \mathrm{d}M = \sum x_2$. For every object k such that $k \in \mathrm{dom}\, x$ holds $x(k) = (r_1 - r_2)(k)$. □

Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partial function f from A to $\mathbb{R}$, and a partition D of A. Now we state the propositions:

(19) Suppose f is bounded and $A \subseteq \mathrm{dom}\, f$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, F = \mathrm{dom}\, D$ and $\bigcup \mathrm{rng}\, F = A$ and for every natural number k such that $k \in \mathrm{dom}\, F$ holds if $\mathrm{len}\, D = 1$, then $F(k) = [\inf A, \sup A]$ and if $\mathrm{len}\, D \neq 1$, then if $k = 1$, then $F(k) = [\inf A, D(k)[$ and if $1 < k < \mathrm{len}\, D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \mathrm{len}\, D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and for every real number x such that $x \in \mathrm{dom}\, g$ there exists a natural number k such that $1 \leqslant k \leqslant \mathrm{len}\, F$ and $x \in F(k)$ and $g(x) = \inf \mathrm{rng}(f \restriction \mathrm{divset}(D, k))$ and $\mathrm{dom}\, g = A$ and $\int g\, \mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, D)$ and for every real number x such that $x \in A$ holds $\inf \mathrm{rng}\, f \leqslant g(x) \leqslant f(x)$.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Consider F being a finite sequence of separated subsets of the Borel sets, g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, F = \mathrm{dom}\, D$ and $\bigcup \mathrm{rng}\, F = A$ and for every natural number k such that $k \in \mathrm{dom}\, F$ holds if $\mathrm{len}\, D = 1$, then $F(k) = [a, b]$ and if $\mathrm{len}\, D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \mathrm{len}\, D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \mathrm{len}\, D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Define $\mathcal{H}$[natural number, extended real] $\equiv \$_2 = \inf \operatorname{rng}(f \restriction \operatorname{divset}(D, \$_1))$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{H}[k, r]$.

Consider h being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} h = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{H}[k, h(k)]$. For every natural number k such that $k \in \operatorname{dom} F$ for every object x such that $x \in F(k)$ holds $g(x) = h(k)$. Define $\mathcal{Z}$[natural number, extended real] $\equiv \$_2 = h(\$_1) \cdot ((\text{B-Meas}) \cdot F)(\$_1)$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{Z}[k, r]$. Consider z being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} z = \operatorname{Seg} \operatorname{len} F$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{Z}[k, z(k)]$. $\int g \, \mathrm{d}\,\text{B-Meas} = \sum z$. For every object p such that $p \in \operatorname{dom} z$ holds $z(p) = (\text{lower\_volume}(f, D))(p)$. For every real number x such that $x \in A$ holds $\inf \operatorname{rng} f \leqslant g(x) \leqslant f(x)$. $\square$

(20) Suppose f is bounded and $A \subseteq \operatorname{dom} f$. Then there exists a finite sequence F of separated subsets of the Borel sets and there exists a partial function g from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [\inf A, \sup A]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [\inf A, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$ and $\operatorname{dom} g = A$ and $\int g \, \mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, D)$ and for every real number x such that $x \in A$ holds $\sup \operatorname{rng} f \geqslant g(x) \geqslant f(x)$.

Proof: Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Consider F being a finite sequence of separated subsets of the Borel sets, g being a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $\operatorname{dom} F = \operatorname{dom} D$ and $\bigcup \operatorname{rng} F = A$ and for every natural number k such that $k \in \operatorname{dom} F$ holds if $\operatorname{len} D = 1$, then $F(k) = [a, b]$ and if $\operatorname{len} D \neq 1$, then if $k = 1$, then $F(k) = [a, D(k)[$ and if $1 < k < \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)[$ and if $k = \operatorname{len} D$, then $F(k) = [D(k -' 1), D(k)]$ and g is simple function in the Borel sets and $\operatorname{dom} g = A$ and for every real number x such that $x \in \operatorname{dom} g$ there exists a natural number k such that $1 \leqslant k \leqslant \operatorname{len} F$ and $x \in F(k)$ and $g(x) = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, k))$. Define $\mathcal{H}$[natural number, extended real] $\equiv \$_2 = \sup \operatorname{rng}(f \restriction \operatorname{divset}(D, \$_1))$ and $\$_2$ is a real number. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{H}[k, r]$.

Consider h being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, h = \mathrm{Seg}\,\mathrm{len}\, F$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $\mathcal{H}[k, h(k)]$. For every natural number k such that $k \in \mathrm{dom}\, F$ for every object x such that $x \in F(k)$ holds $g(x) = h(k)$. Define $\mathcal{Z}[\text{natural number}, \text{extended real}] \equiv \$_2 = h(\$_1) \cdot (\text{B-Meas} \cdot F)(\$_1)$ and $\$_2$ is a real number. For every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ there exists an element r of $\overline{\mathbb{R}}$ such that $\mathcal{Z}[k, r]$. Consider z being a finite sequence of elements of $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, z = \mathrm{Seg}\,\mathrm{len}\, F$ and for every natural number k such that $k \in \mathrm{Seg}\,\mathrm{len}\, F$ holds $\mathcal{Z}[k, z(k)]$. $\int g \,\mathrm{d}\,\text{B-Meas} = \sum z$. For every object p such that $p \in \mathrm{dom}\, z$ holds $z(p) = \text{upper\_volume}(f, D)(p)$. For every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant g(x) \geqslant f(x)$. $\square$

Let us consider a non empty, closed interval subset A of $\mathbb{R}$ and a partial function f from A to $\mathbb{R}$. Now we state the propositions:

(21) Suppose f is bounded and $A \subseteq \mathrm{dom}\, f$ and $\mathrm{vol}(A) > 0$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$ with the same dom and there exists a sequence I of extended reals such that $A = \mathrm{dom}(F(0))$ and for every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n) \,\mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\inf \mathrm{rng}\, f \leqslant F(n)(x) \leqslant f(x)$ and for every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \leqslant F(m)(x)$ and for every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \sup(F\#x)$ and $\sup(F\#x) \leqslant f(x)$ and $\lim F$ is integrable on B-Meas and for every natural number n, $I(n) = \int F(n) \,\mathrm{d}\,\text{B-Meas}$ and I is convergent and $\lim I = \int \lim F \,\mathrm{d}\,\text{B-Meas}$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{partial function from } \mathbb{R} \text{ to } \overline{\mathbb{R}}] \equiv A = \mathrm{dom}\, \$_2$ and $\$_2$ is simple function in the Borel sets and $\int \$_2 \,\mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \mathrm{EqDiv}(A, 2^{\$_1}))$ and for every real number x such that $x \in A$ holds $\inf \mathrm{rng}\, f \leqslant \$_2(x) \leqslant f(x)$ and there exists a finite sequence K of separated subsets of the Borel sets such that $\mathrm{dom}\, K = \mathrm{dom}(\mathrm{EqDiv}(A, 2^{\$_1}))$ and $\bigcup \mathrm{rng}\, K = A$.

For every natural number k such that $k \in \mathrm{dom}\, K$ holds if $\mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1}) = 1$, then $K(k) = [\inf A, \sup A]$ and if $\mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1}) \neq 1$, then if $k = 1$, then $K(k) = [\inf A, (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $1 < k < \mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $k = \mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)]$ and for every real number x such that $x \in \mathrm{dom}\, \$_2$ there exists a natural number k such that $1 \leqslant k \leqslant \mathrm{len}\, K$ and $x \in K(k)$ and $\$_2(x) = \inf \mathrm{rng}(f \restriction \mathrm{divset}(\mathrm{EqDiv}(A, 2^{\$_1}), k))$. For every element n of $\mathbb{N}$, there exists an element g of $\mathbb{R} \dot{\to} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, g]$.

Consider F being a function from $\mathbb{N}$ into $\mathbb{R} \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every natural numbers n, m, $\mathrm{dom}(F(n)) = \mathrm{dom}(F(m))$. For every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{lower\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\inf \mathrm{rng}\, f \leqslant F(n)(x) \leqslant f(x)$. For every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \leqslant F(m)(x)$. For every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \sup(F\#x)$ and $\sup(F\#x) \leqslant f(x)$. Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Reconsider $K = \max(|\inf \mathrm{rng}\, f|, |\sup \mathrm{rng}\, f|)$ as a real number. For every natural number n and for every set x such that $x \in \mathrm{dom}(F(0))$ holds $|F(n)(x)| \leqslant K$. $\square$

(22) Suppose f is bounded and $A \subseteq \mathrm{dom}\, f$ and $\mathrm{vol}(A) > 0$. Then there exists a sequence F of partial functions from $\mathbb{R}$ into $\overline{\mathbb{R}}$ with the same dom and there exists a sequence I of extended reals such that $A = \mathrm{dom}(F(0))$ and for every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \mathrm{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant F(n)(x) \geqslant f(x)$ and for every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \geqslant F(m)(x)$ and for every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \inf(F\#x)$ and $\inf(F\#x) \geqslant f(x)$ and $\lim F$ is integrable on B-Meas and for every natural number n, $I(n) = \int F(n)\,\mathrm{d}\,\text{B-Meas}$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}\,\text{B-Meas}$.

PROOF: Define $\mathcal{P}[\text{natural number}, \text{partial function from } \mathbb{R} \text{ to } \overline{\mathbb{R}}] \equiv A = \mathrm{dom}\,\$_2$ and $\$_2$ is simple function in the Borel sets and $\int \$_2\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \mathrm{EqDiv}(A, 2^{\$_1}))$ and for every real number x such that $x \in A$ holds $\sup \mathrm{rng}\, f \geqslant \$_2(x) \geqslant f(x)$ and there exists a finite sequence K of separated subsets of the Borel sets such that $\mathrm{dom}\, K = \mathrm{dom}(\mathrm{EqDiv}(A, 2^{\$_1}))$ and $\bigcup \mathrm{rng}\, K = A$.

For every natural number k such that $k \in \mathrm{dom}\, K$ holds if $\mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1}) = 1$, then $K(k) = [\inf A, \sup A]$ and if $\mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1}) \neq 1$, then if $k = 1$, then $K(k) = [\inf A, (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $1 < k < \mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)[$ and if $k = \mathrm{len}\,\mathrm{EqDiv}(A, 2^{\$_1})$, then $K(k) = [(\mathrm{EqDiv}(A, 2^{\$_1}))(k -' 1), (\mathrm{EqDiv}(A, 2^{\$_1}))(k)]$ and for every real number x such that $x \in \mathrm{dom}\,\$_2$ there exists a natural number k such that $1 \leqslant k \leqslant \mathrm{len}\, K$ and $x \in K(k)$ and $\$_2(x) = \sup \mathrm{rng}(f \restriction \mathrm{divset}(\mathrm{EqDiv}(A, 2^{\$_1}), k))$.

For every element n of $\mathbb{N}$, there exists an element g of $\mathbb{R} \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, g]$. Consider F being a function from $\mathbb{N}$ into $\mathbb{R} \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every natural numbers n, m, dom$(F(n))$ = dom$(F(m))$. For every natural number n, $F(n)$ is simple function in the Borel sets and $\int F(n)\,\mathrm{d}\,\text{B-Meas} = \text{upper\_sum}(f, \text{EqDiv}(A, 2^n))$ and for every real number x such that $x \in A$ holds $\sup \operatorname{rng} f \geqslant F(n)(x) \geqslant f(x)$. For every natural numbers n, m such that $n \leqslant m$ for every element x of $\mathbb{R}$ such that $x \in A$ holds $F(n)(x) \geqslant F(m)(x)$. For every element x of $\mathbb{R}$ such that $x \in A$ holds $F\#x$ is convergent and $\lim(F\#x) = \inf(F\#x)$ and $\inf(F\#x) \geqslant f(x)$ by [7, (7),(36)]. Consider a, b being real numbers such that $a \leqslant b$ and $A = [a, b]$. Set $K = \max(|\inf \operatorname{rng} f|, |\sup \operatorname{rng} f|)$. For every natural number n and for every set x such that $x \in \text{dom}(F(0))$ holds $|F(n)(x)| \leqslant K$. □

2. Properties of Complete Measure Space

Now we state the propositions:

(23) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, an element E of S, and a natural number n. Suppose $E = \operatorname{dom} f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $M(E \cap \text{GTE-dom}(f, \frac{1}{n+1})) = 0$.

(24) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $E = \operatorname{dom} f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $M(E \cap \text{GT-dom}(f, 0)) = 0$.
PROOF: Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = E \cap \text{GTE-dom}(f, \frac{1}{\$_1+1})$. For every element n of $\mathbb{N}$, there exists an element y of S such that $\mathcal{P}[n, y]$. Consider F being a function from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$. For every element n of $\mathbb{N}$, $(M \cdot F)(n) = 0$. □

(25) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\mathbb{R}$, and an element E of S. Suppose $E = \operatorname{dom} f$ and f is non-negative and E-measurable and $\int f\,\mathrm{d}M = 0$. Then $f =_{\text{a.e.}}^{M} (X \longmapsto 0){\restriction}E$. The theorem is a consequence of (24).

(26) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\mathbb{R}$, and an element E_1 of S. Suppose M is complete and f is E_1-measurable and $f =_{\text{a.e.}}^{M} g$ and $E_1 = \operatorname{dom} f$. Then g is E_1-measurable.
PROOF: Consider E being an element of S such that $M(E) = 0$ and $f{\restriction}E^{\mathrm{c}} = g{\restriction}E^{\mathrm{c}}$. For every real number r, $E_1 \cap \text{LE-dom}(\overline{\mathbb{R}}(g), r) \in S$. □

(27) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then every element of S is an element of $\text{COM}(S, M)$.

(28) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and partial functions f, g from X to $\mathbb{R}$. If $f =^{M}_{\text{a.e.}} g$, then $f =^{\text{COM}(M)}_{\text{a.e.}} g$. The theorem is a consequence of (27).

(29) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f =^{\text{B-Meas}}_{\text{a.e.}} g$. Then $f =^{\text{L-Meas}}_{\text{a.e.}} g$.

(30) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E_1 of S, an element E_2 of $\text{COM}(S, M)$, and a partial function f from X to $\overline{\mathbb{R}}$. If $E_1 = E_2$ and f is E_1-measurable, then f is E_2-measurable. The theorem is a consequence of (27).

(31) Let us consider an element E_1 of the Borel sets, an element E_2 of L-Field, and a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. If $E_1 = E_2$ and f is E_1-measurable, then f is E_2-measurable.

(32) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then every finite sequence of separated subsets of S is a finite sequence of separated subsets of $\text{COM}(S, M)$. The theorem is a consequence of (27).

(33) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. If f is simple function in S, then f is simple function in $\text{COM}(S, M)$. The theorem is a consequence of (32).

(34) Let us consider a set X, a σ-field S of subsets of X, and a σ-measure M on S. Then $\emptyset$ is a set with measure zero w.r.t. M.

(35) Let us consider a set X, a σ-field S of subsets of X, a σ-measure M on S, and an element E of S. Then $M(E) = \text{COM}(M)(E)$. The theorem is a consequence of (34).

(36) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int\limits_{f} M(x)dx = \int\limits_{f} \text{COM}(M)(x)dx$.

Proof: Consider F being a finite sequence of separated subsets of S, a, x being finite sequences of elements of $\overline{\mathbb{R}}$ such that F and a are representation of f and $a(1) = 0_{\overline{\mathbb{R}}}$ and for every natural number n such that $2 \leqslant n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n) < +\infty$ and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$ and $\int\limits_{f} M(x)dx = \sum x$. f is simple function in $\text{COM}(S, M)$. Reconsider $F_1 = F$ as a finite sequence of separated subsets of $\text{COM}(S, M)$. For every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (\text{COM}(M) \cdot$

$F_1)(n)$. □

(37) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $E = \operatorname{dom} f$ and f is E-measurable and non-negative. Then $\int^+ f \, \mathrm{d}M = \int^+ f \, \mathrm{d}\mathrm{COM}(M)$.

PROOF: Consider F being a sequence of partial functions from X into $\overline{\mathbb{R}}$ such that for every natural number n, $F(n)$ is simple function in S and $\operatorname{dom}(F(n)) = \operatorname{dom} f$ and for every natural number n, $F(n)$ is non-negative and for every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leqslant F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$. Reconsider $g = F(0)$ as a partial function from X to $\overline{\mathbb{R}}$. For every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leqslant \lim(F\#x)$.

Consider K being a sequence of extended reals such that for every natural number n, $K(n) = \int' F(n) \, \mathrm{d}M$ and K is convergent and $\sup \operatorname{rng} K = \lim K$ and $\int' g \, \mathrm{d}M \leqslant \lim K$. Reconsider $E_1 = E$ as an element of $\mathrm{COM}(S, M)$. f is E_1-measurable. For every natural number n, $F(n)$ is simple function in $\mathrm{COM}(S, M)$ and $\operatorname{dom}(F(n)) = \operatorname{dom} f$. For every natural number n, $K(n) = \int' F(n) \, \mathrm{d}\mathrm{COM}(M)$. □

(38) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then

(i) f is integrable on $\mathrm{COM}(M)$, and

(ii) $\int f \, \mathrm{d}M = \int f \, \mathrm{d}\mathrm{COM}(M)$.

The theorem is a consequence of (27), (37), and (30).

3. Relation Between Riemann and Lebesgue Integrals

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and partial functions f, g from X to $\mathbb{R}$. Now we state the propositions:

(39) If ($E = \operatorname{dom} f$ or $E = \operatorname{dom} g$) and $f =^{M}_{\text{a.e.}} g$, then $f - g =^{M}_{\text{a.e.}} (X \longmapsto 0) \restriction E$.

PROOF: Consider A being an element of S such that $M(A) = 0$ and $f \restriction A^{\mathrm{c}} = g \restriction A^{\mathrm{c}}$. For every element x of X such that $x \in \operatorname{dom}((f - g) \restriction A^{\mathrm{c}})$ holds $((f - g) \restriction A^{\mathrm{c}})(x) = (((X \longmapsto 0) \restriction E) \restriction A^{\mathrm{c}})(x)$. □

(40) If $E = \operatorname{dom}(f - g)$ and $f - g =^{M}_{\text{a.e.}} (X \longmapsto 0) \restriction E$, then $f \restriction E =^{M}_{\text{a.e.}} g \restriction E$.

Proof: Consider A being an element of S such that $M(A) = 0$ and $(f - g)\restriction A^{c} = ((X \longmapsto 0)\restriction E)\restriction A^{c}$. For every element x of X such that $x \in \mathrm{dom}((f\restriction E)\restriction A^{c})$ holds $((f\restriction E)\restriction A^{c})(x) = ((g\restriction E)\restriction A^{c})(x)$. $\square$

(41) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a partial function f from X to $\mathbb{R}$. Suppose $E = \mathrm{dom}\, f$ and $M(E) < +\infty$ and f is bounded and E-measurable. Then f is integrable on M.

(42) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and partial functions f, g from X to $\mathbb{R}$. Then $f =_{\text{a.e.}}^{M} g$ if and only if $\max_{+}(f) =_{\text{a.e.}}^{M} \max_{+}(g)$ and $\max_{-}(f) =_{\text{a.e.}}^{M} \max_{-}(g)$.
Proof: Consider E_1 being an element of S such that $M(E_1) = 0$ and $\max_{+}(f)\restriction {E_1}^{c} = \max_{+}(g)\restriction {E_1}^{c}$. Consider E_2 being an element of S such that $M(E_2) = 0$ and $\max_{-}(f)\restriction {E_2}^{c} = \max_{-}(g)\restriction {E_2}^{c}$. Set $E = E_1 \cup E_2$. For every element x of X such that $x \in \mathrm{dom}(f\restriction E^{c})$ holds $(f\restriction E^{c})(x) = (g\restriction E^{c})(x)$. $\square$

(43) Let us consider a non empty set X, and a partial function f from X to $\mathbb{R}$. Then

(i) $\max_{+}(\overline{\mathbb{R}}(f)) = \overline{\mathbb{R}}(\max_{+}(f))$, and

(ii) $\max_{-}(\overline{\mathbb{R}}(f)) = \overline{\mathbb{R}}(\max_{-}(f))$.

(44) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, partial functions f, g from X to $\mathbb{R}$, and an element E of S. Suppose M is complete and f is integrable on M and $f =_{\text{a.e.}}^{M} g$ and $E = \mathrm{dom}\, f$ and $E = \mathrm{dom}\, g$. Then

(i) g is integrable on M, and

(ii) $\int f \,\mathrm{d}M = \int g \,\mathrm{d}M$.

The theorem is a consequence of (26), (43), and (42).

(45) Let us consider a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$, and a real number a. Suppose $a \in \mathrm{dom}\, f$. Then there exists an element A of the Borel sets such that

(i) $A = \{a\}$, and

(ii) f is A-measurable, and

(iii) $f\restriction A$ is integrable on B-Meas, and

(iv) $\int f\restriction A \,\mathrm{d}\,\text{B-Meas} = 0$.

(46) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $a \in \mathrm{dom}\, f$. Then there exists an element A of the Borel sets such that

(i) $A = \{a\}$, and

(ii) f is A-measurable, and

(iii) $f \restriction A$ is integrable on B-Meas, and

(iv) $\int f \restriction A \, \mathrm{d}\, \text{B-Meas} = 0$.

The theorem is a consequence of (45).

(47) Let us consider a partial function f from $\mathbb{R}$ to $\overline{\mathbb{R}}$. Suppose f is integrable on B-Meas. Then

(i) f is integrable on L-Meas, and

(ii) $\int f \, \mathrm{d}\, \text{B-Meas} = \int f \, \mathrm{d}\, \text{L-Meas}$.

(48) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is integrable on B-Meas. Then

(i) f is integrable on L-Meas, and

(ii) $\int f \, \mathrm{d}\, \text{B-Meas} = \int f \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (38).

(49) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, an element A_1 of L-Field, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $A = A_1$ and $A \subseteq \operatorname{dom} f$ and $f \restriction A$ is bounded and f is integrable on A. Then

(i) f is A_1-measurable, and

(ii) $f \restriction A_1$ is integrable on L-Meas, and

(iii) $\operatorname{integral} f \restriction A = \int f \restriction A \, \mathrm{d}\, \text{L-Meas}$.

The theorem is a consequence of (46), (30), (48), (21), (22), (17), (3), (25), (29), (40), (26), (41), (38), and (44).

(50) Let us consider real numbers a, b, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$ and $f \restriction [a, b]$ is bounded and f is integrable on $[a, b]$. Then $\int\limits_a^b f(x)dx = \int f \restriction [a, b] \, \mathrm{d}\, \text{L-Meas}$. The theorem is a consequence of (49).

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