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# Duality Notions in Real Projective Plane[1]

Roland Coghetto
cafr-MSA2P asbl
Rue de la Brasserie 5
7100 La Louvière, Belgium

**Summary.** In this article, we check with the Mizar system [1], [2], the converse of Desargues' theorem and the converse of Pappus' theorem of the real projective plane. It is well known that in the projective plane, the notions of points and lines are dual [11], [9], [15], [8]. Some results (analytical, synthetic, combinatorial) of projective geometry are already present in some libraries Lean/Hott [5], Isabelle/Hol [3], Coq [13], [14], [4], Agda [6], . . . .

Proofs of dual statements by proof assistants have already been proposed, using an axiomatic method (for example see in [13] - the section on duality: "[...] For every theorem we prove, we can easily derive its dual using our function `swap` [...][2]").

In our formalisation, we use an analytical rather than a synthetic approach using the definitions of Leończuk and Prażmowski of the projective plane [12]. Our motivation is to show that it is possible by developing dual definitions to find proofs of dual theorems in a few lines of code.

In the first part, rather technical, we introduce definitions that allow us to construct the duality between the points of the real projective plane and the lines associated to this projective plane. In the second part, we give a natural definition of line concurrency and prove that this definition is dual to the definition of alignment. Finally, we apply these results to find, in a few lines, the dual properties and theorems of those defined in the article [12] (`transitive`, `Vebleian`, `at_least_3rank`, `Fanoian`, `Desarguesian`, `2-dimensional`).

We hope that this methodology will allow us to continued more quickly the proof started in [7] that the Beltrami-Klein plane is a model satisfying the axioms of the hyperbolic plane (in the sense of Tarski geometry [10]).

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[2]https://github.com/coq-contribs/projective-geometry




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## 1. Preliminaries

Now we state the proposition:

(1) Let us consider real numbers $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$. Then $\langle|[a, b, c], [d, e, f], [g, h, i]|\rangle = a \cdot e \cdot i + b \cdot f \cdot g + c \cdot d \cdot h - g \cdot e \cdot c - h \cdot f \cdot a - i \cdot d \cdot b$.

Let us consider real numbers $a$, $b$, $c$, $d$, $e$. Now we state the propositions:

(2) $\langle|[a, 1, 0], [b, 0, 1], [c, d, e]|\rangle = c - a \cdot d - e \cdot b$.

(3) $\langle|[1, a, 0], [0, b, 1], [c, d, e]|\rangle = b \cdot e + a \cdot c - d$.

(4) $\langle|[1, 0, a], [0, 1, b], [c, d, e]|\rangle = e - c \cdot a - d \cdot b$.

(5) Let us consider an element $u$ of $\mathcal{E}^3_{\rm T}$. Then $u$ is zero if and only if $|(u, u)| = 0$.

Let us consider non zero elements $u$, $v$, $w$ of $\mathcal{E}^3_{\rm T}$. Now we state the propositions:

(6) If $\langle|u, v, w|\rangle = 0$, then there exists a non zero element $p$ of $\mathcal{E}^3_{\rm T}$ such that $|(p, u)| = 0$ and $|(p, v)| = 0$ and $|(p, w)| = 0$.

(7) If $|(u, v)| = 0$ and $w$ and $v$ are proportional, then $|(u, w)| = 0$.

(8) Let us consider non zero elements $a$, $u$, $v$ of $\mathcal{E}^3_{\rm T}$. Suppose $u$ and $v$ are not proportional and $|(a, u)| = 0$ and $|(a, v)| = 0$. Then $a$ and $u \times v$ are proportional.

(9) Let us consider non zero elements $u$, $v$ of $\mathcal{E}^3_{\rm T}$, and a real number $r$. If $r \neq 0$ and $u$ and $v$ are proportional, then $r \cdot u$ and $v$ are proportional.

## 2. Dual of a Point - Dual of a Line

Let $P$ be a point of the projective space over $\mathcal{E}^3_{\rm T}$. We say that $P$ is $\pi_1$-zero if and only if

(Def. 1) for every non zero element $u$ of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of $u$ holds $u(1) = 0$.

Note that there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is $\pi_1$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non $\pi_1$-zero.

Now we state the proposition:

(10) Let us consider a non $\pi_1$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element $u$ of $\mathcal{E}^3_{\rm T}$. If $P =$ the direction of $u$, then $u(1) \neq 0$.

Let $P$ be a non $\pi_1$-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\widetilde{\pi_1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by

(Def. 2) the direction of $it = P$ and $it(1) = 1$.

Now we state the propositions:

(11) Let us consider a non $\pi_1$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element $u$ of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of $u$. Then $\widetilde{\pi_1}(P) = [1, \frac{u(2)}{u(1)}, \frac{u(3)}{u(1)}]$.

(12) Let us consider a non $\pi_1$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a point $Q$ of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $Q =$ the direction of $\widetilde{\pi_1}(P)$. Then $Q$ is not $\pi_1$-zero.

Let $P$ be a point of the projective space over $\mathcal{E}^3_{\rm T}$. We say that $P$ is $\pi_2$-zero if and only if

(Def. 3) for every non zero element $u$ of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of $u$ holds $u(2) = 0$.

One can verify that there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is $\pi_2$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non $\pi_2$-zero.

Now we state the proposition:

(13) Let us consider a non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element $u$ of $\mathcal{E}^3_{\rm T}$. If $P =$ the direction of $u$, then $u(2) \neq 0$.

Let $P$ be a non $\pi_2$-zero point of the projective space over $\mathcal{E}^3_{\rm T}$. The functor $\widetilde{\pi_2}(P)$ yielding a non zero element of $\mathcal{E}^3_{\rm T}$ is defined by

(Def. 4) the direction of $it = P$ and $it(2) = 1$.

Now we state the propositions:

(14) Let us consider a non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a non zero element $u$ of $\mathcal{E}^3_{\rm T}$. Suppose $P =$ the direction of $u$. Then $\widetilde{\pi_2}(P) = [\frac{u(1)}{u(2)}, 1, \frac{u(3)}{u(2)}]$.

(15) Let us consider a non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\rm T}$, and a point $Q$ of the projective space over $\mathcal{E}^3_{\rm T}$. Suppose $Q =$ the direction of $\widetilde{\pi_2}(P)$. Then $Q$ is not $\pi_2$-zero.

Let $P$ be a point of the projective space over $\mathcal{E}^3_{\rm T}$. We say that $P$ is $\pi_3$-zero if and only if

(Def. 5) for every non zero element $u$ of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of $u$ holds $u(3) = 0$.

Observe that there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is $\pi_3$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\rm T}$ which is non $\pi_3$-zero.

Now we state the proposition:

(16) Let us consider a non $\pi_3$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. If $P =$ the direction of $u$, then $u(3) \neq 0$.

Let $P$ be a non $\pi_3$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\widetilde{\pi_3}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 6) the direction of $it = P$ and $it(3) = 1$.

Now we state the propositions:

(17) Let us consider a non $\pi_3$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of $u$. Then $\widetilde{\pi_3}(P) = [\frac{u(1)}{u(3)}, \frac{u(2)}{u(3)}, 1]$.

(18) Let us consider a non $\pi_3$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a point $Q$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $Q =$ the direction of $\widetilde{\pi_3}(P)$. Then $Q$ is not $\pi_3$-zero.

Let us observe that there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non $\pi_1$-zero and non $\pi_2$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non $\pi_1$-zero and non $\pi_3$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non $\pi_2$-zero and non $\pi_3$-zero and there exists a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ which is non $\pi_1$-zero, non $\pi_2$-zero, and non $\pi_3$-zero.

Let $P$ be a non $\pi_1$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 7) $[-(\widetilde{\pi_1}(P))(2), 1, 0]$.

The functor $\mathrm{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 8) the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P)$.

The functor $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 9) $[-(\widetilde{\pi_1}(P))(3), 0, 1]$.

The functor $\mathrm{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 10) the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

Let us consider a non $\pi_1$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(19) $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P) \neq \mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

(20) The direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_2,1,0}(P) \neq$ the direction of $\mathrm{dir}_{(-\widetilde{\pi_1})_3,0,1}(P)$.

(21) Let us consider a non $\pi_1$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$, and an element $v$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u =$

$\widetilde{\pi_1}(P)$. Then $\langle|\operatorname{dir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{dir}_{(-\widetilde{\pi_1})_3,0,1}(P), v|\rangle = |(u,v)|$. The theorem is a consequence of (11) and (2).

(22) Let us consider a non $\pi_1$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_1}(P)$. Then $\langle|\operatorname{dir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{dir}_{(-\widetilde{\pi_1})_3,0,1}(P), \widetilde{\pi_1}(P)|\rangle = 1 + u(2)\cdot u(2) + u(3)\cdot u(3)$. The theorem is a consequence of (21).

Let $P$ be a non $\pi_2$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 11) $[1, -(\widetilde{\pi_2}(P))(1), 0]$.

The functor $\operatorname{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 12) the direction of $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P)$.

The functor $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 13) $[0, -(\widetilde{\pi_2}(P))(3), 1]$.

The functor $\operatorname{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 14) the direction of $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

Let us consider a non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(23) $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P) \neq \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

(24) The direction of $\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P) \neq$ the direction of $\operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P)$.

(25) Let us consider a non $\pi_2$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$, and an element $v$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_2}(P)$. Then $\langle|\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P), v|\rangle = -|(u,v)|$. The theorem is a consequence of (14) and (3).

(26) Let us consider a non $\pi_2$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_2}(P)$. Then $\langle|\operatorname{dir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{dir}_{0,(-\widetilde{\pi_2})_3,1}(P), \widetilde{\pi_2}(P)|\rangle = -(u(1)\cdot u(1) + 1 + u(3)\cdot u(3))$. The theorem is a consequence of (25).

Let $P$ be a non $\pi_3$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\operatorname{dir}_{1,0,(-\widetilde{\pi_3})_1}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 15) $[1, 0, -(\widetilde{\pi_3}(P))(1)]$.

The functor $\operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P)$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 16) the direction of $\operatorname{dir}_{1,0,(-\widetilde{\pi_3})_1}(P)$.

The functor $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$ yielding a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 17) $[0, 1, -(\widetilde{\pi_3}(P))(2)]$.

The functor $\mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P$ yielding a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 18) the direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

Let us consider a non $\pi_3$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(27) $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P) \neq \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

(28) The direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P) \neq$ the direction of $\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

(29) Let us consider a non $\pi_3$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$, and an element $v$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_3}(P)$. Then $\langle|\,\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P), \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P), v|\rangle = |(u, v)|$. The theorem is a consequence of (17) and (4).

(30) Let us consider a non $\pi_3$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = \widetilde{\pi_3}(P)$. Then $\langle|\,\mathrm{dir}_{1,0,(-\widetilde{\pi_3})_1}(P), \mathrm{dir}_{1,0,(-\widetilde{\pi_3})_2}(P), \widetilde{\pi_3}(P)|\rangle = u(1) \cdot u(1) + u(2) \cdot u(2) + 1$. The theorem is a consequence of (29).

Let $P$ be a non $\pi_1$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_1(P)$ yielding an element of $L$(the real projective plane) is defined by the term

(Def. 19) $\mathrm{Line}(\mathrm{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P), \mathrm{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P))$.

Let $P$ be a non $\pi_2$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_2(P)$ yielding an element of $L$(the real projective plane) is defined by the term

(Def. 20) $\mathrm{Line}(\mathrm{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P), \mathrm{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P))$.

Let $P$ be a non $\pi_3$-zero point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{dual}_3(P)$ yielding an element of $L$(the real projective plane) is defined by the term

(Def. 21) $\mathrm{Line}(\mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P), \mathrm{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P)$.

Let us consider a non $\pi_1$-zero, non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and a non zero element $u$ of $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(31) Suppose $P =$ the direction of $u$. Then

(i) $\widetilde{\pi_1}(P) = [1, \frac{u(2)}{u(1)}, \frac{u(3)}{u(1)}]$, and

(ii) $\widetilde{\pi_2}(P) = [\frac{u(1)}{u(2)}, 1, \frac{u(3)}{u(2)}]$.

(32) Suppose $P =$ the direction of $u$. Then

(i) $\widetilde{\pi_1}(P) = \frac{u(2)}{u(1)} \cdot (\widetilde{\pi_2}(P))$, and

(ii) $\widetilde{\pi_2}(P) = \frac{u(1)}{u(2)} \cdot (\widetilde{\pi_1}(P))$.

The theorem is a consequence of (10), (13), (11), and (14).

Let us consider a non $\pi_1$-zero, non $\pi_2$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(33) $\mathrm{dual}_1(P) = \mathrm{dual}_2(P)$. The theorem is a consequence of (11), (14), (2), (10), (3), and (13).

(34) $\mathrm{dual}_2(P) = \mathrm{dual}_3(P)$. The theorem is a consequence of (17), (14), (3), (13), (16), and (4).

(35) $\mathrm{dual}_1(P) = \mathrm{dual}_3(P)$. The theorem is a consequence of (11), (17), (2), (10), (4), and (16).

(36) Let us consider a non $\pi_1$-zero, non $\pi_2$-zero, non $\pi_3$-zero point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Then

(i) $\mathrm{dual}_1(P) = \mathrm{dual}_2(P)$, and

(ii) $\mathrm{dual}_1(P) = \mathrm{dual}_3(P)$, and

(iii) $\mathrm{dual}_2(P) = \mathrm{dual}_3(P)$.

(37) Every element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is non $\pi_1$-zero or non $\pi_2$-zero or non $\pi_3$-zero non $\pi_1$-zero non $\pi_2$-zero or non $\pi_3$-zero.

Let $P$ be a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. The functor dual $P$ yielding an element of $L$(the real projective plane) is defined by

(Def. 22) (i) there exists a non $\pi_1$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and *it* $= \mathrm{dual}_1(P')$, **if** $P$ is not $\pi_1$-zero,

(ii) there exists a non $\pi_2$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and *it* $= \mathrm{dual}_2(P')$, **if** $P$ is $\pi_1$-zero and non $\pi_2$-zero,

(iii) there exists a non $\pi_3$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P' = P$ and *it* $= \mathrm{dual}_3(P')$, **if** $P$ is $\pi_1$-zero, $\pi_2$-zero, and non $\pi_3$-zero.

Let $P$ be a point of the real projective plane. The functor $\# P$ yielding an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 23) $P$.

The functor dual $P$ yielding an element of $L$(the real projective plane) is defined by the term

(Def. 24) dual $\# P$.

Let us consider an element $P$ of the real projective plane. Now we state the propositions:

(38) Suppose $\# P$ is not $\pi_1$-zero. Then there exists a non $\pi_1$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

 (i) $P = P'$, and

 (ii) $\operatorname{dual} P = \operatorname{dual}_1(P')$.

(39) Suppose $\# P$ is not $\pi_2$-zero. Then there exists a non $\pi_2$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

 (i) $P = P'$, and

 (ii) $\operatorname{dual} P = \operatorname{dual}_2(P')$.

The theorem is a consequence of (33).

(40) Suppose $\# P$ is not $\pi_3$-zero. Then there exists a non $\pi_3$-zero point $P'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

 (i) $P = P'$, and

 (ii) $\operatorname{dual} P = \operatorname{dual}_3(P')$.

The theorem is a consequence of (34) and (35).

Let us consider a non $\pi_1$-zero element $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(41) $P \notin \operatorname{Line}(\operatorname{Pdir}_{(-\widetilde{\pi_1})_2,1,0}(P), \operatorname{Pdir}_{(-\widetilde{\pi_1})_3,0,1}(P))$. The theorem is a consequence of (21) and (5).

(42) $P \notin \operatorname{Line}(\operatorname{Pdir}_{1,(-\widetilde{\pi_2})_1,0}(P), \operatorname{Pdir}_{0,(-\widetilde{\pi_2})_3,1}(P))$. The theorem is a consequence of (25) and (5).

(43) $P \notin \operatorname{Line}(\operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_1}(P), \operatorname{Pdir}_{1,0,(-\widetilde{\pi_3})_2}(P)$. The theorem is a consequence of (29) and (5).

(44) Let us consider a point $P$ of the real projective plane. Then $P \notin \operatorname{dual} P$. The theorem is a consequence of (37), (38), (41), (39), (42), (40), and (43).

Let $l$ be an element of $L$(the real projective plane). The functor $\operatorname{dual} l$ yielding a point of the real projective plane is defined by

(Def. 25) there exist points $P$, $Q$ of the real projective plane such that $P \neq Q$ and $l = \operatorname{Line}(P, Q)$ and $it = \operatorname{L2P}(P, Q)$.

Now we state the propositions:

(45) Let us consider a point $P$ of the real projective plane. Then $\operatorname{dual}\operatorname{dual} P = P$. The theorem is a consequence of (37), (38), (11), (10), (8), (9), (39), (14), (13), (40), (17), and (16).

(46) Let us consider an element $l$ of $L$(the real projective plane). Then $\operatorname{dual}\operatorname{dual} l = l$. The theorem is a consequence of (37), (38), (10), (11), (20), (2), (39), (13), (14), (24), (3), (40), (16), (17), (28), and (4).

(47) Let us consider points $P$, $Q$ of the real projective plane. Then $P \neq Q$ if and only if dual $P \neq$ dual $Q$. The theorem is a consequence of (45).

(48) Let us consider elements $l$, $m$ of $L$(the real projective plane). Then $l \neq m$ if and only if dual $l \neq$ dual $m$. The theorem is a consequence of (46).

## 3. Two Dual Notions: Concurrency and Collinearity

Let $l_1$, $l_2$, $l_3$ be elements of $L$(the real projective plane). We say that $l_1$, $l_2$, $l_3$ are concurrent if and only if

(Def. 26) there exists a point $P$ of the real projective plane such that $P \in l_1$ and $P \in l_2$ and $P \in l_3$.

Let $l$ be an element of $L$(the real projective plane). The functor $\# l$ yielding a line of Inc-ProjSp(the real projective plane) is defined by the term

(Def. 27) $l$.

Let $l$ be a line of Inc-ProjSp(the real projective plane). The functor $\# l$ yielding an element of $L$(the real projective plane) is defined by the term

(Def. 28) $l$.

Now we state the propositions:

(49) Let us consider elements $l_1$, $l_2$, $l_3$ of $L$(the real projective plane). Then $l_1$, $l_2$, $l_3$ are concurrent if and only if $\# l_1$, $\# l_2$, $\# l_3$ are concurrent.

(50) Let us consider lines $l_1$, $l_2$, $l_3$ of Inc-ProjSp(the real projective plane). Then $l_1$, $l_2$, $l_3$ are concurrent if and only if $\# l_1$, $\# l_2$, $\# l_3$ are concurrent. The theorem is a consequence of (49).

(51) Let us consider elements $P$, $Q$, $R$ of the real projective plane. Suppose $P$, $Q$ and $R$ are collinear. Then

(i) $Q$, $R$ and $P$ are collinear, and

(ii) $R$, $P$ and $Q$ are collinear, and

(iii) $P$, $R$ and $Q$ are collinear, and

(iv) $R$, $Q$ and $P$ are collinear, and

(v) $Q$, $P$ and $R$ are collinear.

(52) Let us consider elements $l_1$, $l_2$, $l_3$ of $L$(the real projective plane). Suppose $l_1$, $l_2$, $l_3$ are concurrent. Then

(i) $l_2$, $l_1$, $l_3$ are concurrent, and

(ii) $l_1$, $l_3$, $l_2$ are concurrent, and

(iii) $l_3$, $l_2$, $l_1$ are concurrent, and

(iv) $l_3$, $l_2$, $l_1$ are concurrent, and

(v) $l_2$, $l_3$, $l_1$ are concurrent.

(53) Let us consider points $P$, $Q$ of the real projective plane, and elements $P'$, $Q'$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P = P'$ and $Q = Q'$, then $\mathrm{Line}(P, Q) = \mathrm{Line}(P', Q')$.

Let us consider a point $P$ of the real projective plane and an element $l$ of $L$(the real projective plane). Now we state the propositions:

(54) If $P \in l$, then $\mathrm{dual}\, l \in \mathrm{dual}\, P$. The theorem is a consequence of (37), (38), (21), (7), (39), (25), (40), and (29).

(55) If $\mathrm{dual}\, l \in \mathrm{dual}\, P$, then $P \in l$. The theorem is a consequence of (54), (45), and (46).

(56) Let us consider points $P$, $Q$, $R$ of the real projective plane. Suppose $P$, $Q$ and $R$ are collinear. Then $\mathrm{dual}\, P$, $\mathrm{dual}\, Q$, $\mathrm{dual}\, R$ are concurrent. The theorem is a consequence of (54).

(57) Let us consider an element $l$ of $L$(the real projective plane), and points $P$, $Q$, $R$ of the real projective plane. If $P, Q, R \in l$, then $P$, $Q$ and $R$ are collinear.

(58) Let us consider elements $l_1, l_2, l_3$ of $L$(the real projective plane). Suppose $l_1$, $l_2$, $l_3$ are concurrent. Then $\mathrm{dual}\, l_1$, $\mathrm{dual}\, l_2$ and $\mathrm{dual}\, l_3$ are collinear. The theorem is a consequence of (54) and (57).

(59) Let us consider points $P$, $Q$, $R$ of the real projective plane. Then $P$, $Q$ and $R$ are collinear if and only if $\mathrm{dual}\, P$, $\mathrm{dual}\, Q$, $\mathrm{dual}\, R$ are concurrent. The theorem is a consequence of (56), (58), and (45).

(60) Let us consider elements $l_1, l_2, l_3$ of $L$(the real projective plane). Then $l_1$, $l_2$, $l_3$ are concurrent if and only if $\mathrm{dual}\, l_1$, $\mathrm{dual}\, l_2$ and $\mathrm{dual}\, l_3$ are collinear. The theorem is a consequence of (46) and (59).

## 4. Some Dual Properties of a Real Projective Plane

Now we state the propositions:

(61) The real projective plane is reflexive, transitive, Vebleian, at least 3 rank, Fanoian, Desarguesian, Pappian, and 2-dimensional.

(62) Converse reflexive:
Let us consider elements $l$, $m$, $n$ of $L$(the real projective plane). Then

(i) $l$, $m$, $l$ are concurrent, and

(ii) $l$, $l$, $m$ are concurrent, and

(iii) $l$, $m$, $m$ are concurrent.

The theorem is a consequence of (59) and (46).

(63) CONVERSE TRANSITIVE:

Let us consider elements $l$, $m$, $n$, $n_1$, $n_2$ of $L$(the real projective plane). Suppose $l \neq m$ and $l$, $m$, $n$ are concurrent and $l$, $m$, $n_1$ are concurrent and $l$, $m$, $n_2$ are concurrent. Then $n$, $n_1$, $n_2$ are concurrent. The theorem is a consequence of (60), (48), (59), and (46).

(64) CONVERSE VEBLIEAN:

Let us consider elements $l$, $l_1$, $l_2$, $n$, $n_1$ of $L$(the real projective plane). Suppose $l$, $l_1$, $n$ are concurrent and $l_1$, $l_2$, $n_1$ are concurrent. Then there exists an element $n_2$ of $L$(the real projective plane) such that

(i) $l$, $l_2$, $n_2$ are concurrent, and

(ii) $n$, $n_1$, $n_2$ are concurrent.

The theorem is a consequence of (60), (59), and (46).

(65) CONVERSE AT LEAST 3-RANK:

Let us consider elements $l$, $m$ of $L$(the real projective plane). Then there exists an element $n$ of $L$(the real projective plane) such that

(i) $l \neq n$, and

(ii) $m \neq n$, and

(iii) $l$, $m$, $n$ are concurrent.

The theorem is a consequence of (45), (59), and (46).

(66) CONVERSE FANOIAN:

Let us consider elements $l_1$, $n_2$, $m$, $n_1$, $m_1$, $l$, $n$ of $L$(the real projective plane). Suppose $l_1$, $n_2$, $m$ are concurrent and $n_1$, $m_1$, $m$ are concurrent and $l_1$, $n_1$, $l$ are concurrent and $n_2$, $m_1$, $l$ are concurrent and $l_1$, $m_1$, $n$ are concurrent and $n_2$, $n_1$, $n$ are concurrent and $l$, $m$, $n$ are concurrent. Then

(i) $l_1$, $n_2$, $m_1$ are concurrent, or

(ii) $l_1$, $n_2$, $n_1$ are concurrent, or

(iii) $l_1$, $n_1$, $m_1$ are concurrent, or

(iv) $n_2$, $n_1$, $m_1$ are concurrent.

The theorem is a consequence of (60).

(67) CONVERSE DESARGUESIAN:

Let us consider elements $k$, $l_1$, $l_2$, $l_3$, $m_1$, $m_2$, $m_3$, $n_1$, $n_2$, $n_3$ of $L$(the real projective plane). Suppose $k \neq m_1$ and $l_1 \neq m_1$ and $k \neq m_2$ and $l_2 \neq m_2$ and $k \neq m_3$ and $l_3 \neq m_3$ and $k$, $l_1$, $l_2$ are not concurrent and $k$, $l_1$, $l_3$ are not concurrent and $k$, $l_2$, $l_3$ are not concurrent and $l_1$, $l_2$, $n_3$ are concurrent and $m_1$, $m_2$, $n_3$ are concurrent and $l_2$, $l_3$, $n_1$ are concurrent and $m_2$, $m_3$, $n_1$

are concurrent and $l_1$, $l_3$, $n_2$ are concurrent and $m_1$, $m_3$, $n_2$ are concurrent and $k$, $l_1$, $m_1$ are concurrent and $k$, $l_2$, $m_2$ are concurrent and $k$, $l_3$, $m_3$ are concurrent. Then $n_1$, $n_2$, $n_3$ are concurrent. The theorem is a consequence of (48) and (60).

(68) Converse Pappian:
Let us consider elements $k$, $l_1$, $l_2$, $l_3$, $m_1$, $m_2$, $m_3$, $n_1$, $n_2$, $n_3$ of $L$(the real projective plane). Suppose $k \neq l_2$ and $k \neq l_3$ and $l_2 \neq l_3$ and $l_1 \neq l_2$ and $l_1 \neq l_3$ and $k \neq m_2$ and $k \neq m_3$ and $m_2 \neq m_3$ and $m_1 \neq m_2$ and $m_1 \neq m_3$ and $k$, $l_1$, $m_1$ are not concurrent and $k$, $l_1$, $l_2$ are concurrent and $k$, $l_1$, $l_3$ are concurrent and $k$, $m_1$, $m_2$ are concurrent and $k$, $m_1$, $m_3$ are concurrent and $l_1$, $m_2$, $n_3$ are concurrent and $m_1$, $l_2$, $n_3$ are concurrent and $l_1$, $m_3$, $n_2$ are concurrent and $l_3$, $m_1$, $n_2$ are concurrent and $l_2$, $m_3$, $n_1$ are concurrent and $l_3$, $m_2$, $n_1$ are concurrent. Then $n_1$, $n_2$, $n_3$ are concurrent. The theorem is a consequence of (48) and (60).

(69) Converse 2-dimensional:
Let us consider elements $l$, $l_1$, $m$, $m_1$ of $L$(the real projective plane). Then there exists an element $n$ of $L$(the real projective plane) such that

(i) $l$, $l_1$, $n$ are concurrent, and

(ii) $m$, $m_1$, $n$ are concurrent.

The theorem is a consequence of (59) and (46).

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