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Some Properties of Membership Functions Composed of Triangle Functions and Piecewise Linear Functions[1]

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Summary. IF-THEN rules in fuzzy inference is composed of multiple fuzzy sets (membership functions). IF-THEN rules can therefore be considered as a pair of membership functions [7]. The evaluation function of fuzzy control is composite function with fuzzy approximate reasoning and is functional on the set of membership functions. We obtained continuity of the evaluation function and compactness of the set of membership functions [12]. Therefore, we proved the existence of pair of membership functions, which maximizes (minimizes) evaluation function and is considered IF-THEN rules, in the set of membership functions by using extreme value theorem. The set of membership functions (fuzzy sets) is defined in this article to verifier our proofs before by Mizar [9], [10], [4]. Membership functions composed of triangle function, piecewise linear function and Gaussian function used in practice are formalized using existing functions.

On the other hand, not only curve membership functions mentioned above but also membership functions composed of straight lines (piecewise linear function) like triangular and trapezoidal functions are formalized. Moreover, different from the definition in [3] formalizations of triangular and trapezoidal function composed of two straight lines, minimum function and maximum functions are proposed. We prove, using the Mizar [2], [1] formalism, some properties of membership functions such as continuity and periodicity [13], [8].

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1. Preliminaries

Now we state the propositions:

(1) Let us consider real numbers a, b, c, d. Then $|\max(c, \min(d, a)) - \max(c, \min(d, b))| \leqslant |a - b|$.

(2) Let us consider a real number x. Then $|\sin x| \leqslant |x|$.

(3) Let us consider real numbers x, y. Then $|\sin x - \sin y| \leqslant |x - y|$. The theorem is a consequence of (2).

(4) Let us consider a real number x. If $\exp x = 1$, then $x = 0$.

(5) Let us consider real numbers a, b, p, q. Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$. Then

   (i) $\frac{-b}{a} < \frac{q-b}{a+p} < \frac{q}{p}$, and

   (ii) $\frac{a \cdot q + b \cdot p}{a+p} > 0$.

(6) Let us consider real numbers a, b, p, q, s. Suppose $a > 0$ and $p > 0$ and $\frac{s-b}{a} = \frac{s-q}{-p}$. Then

   (i) $\frac{s-b}{a} = \frac{q-b}{a+p}$, and

   (ii) $\frac{s-q}{-p} = \frac{q-b}{a+p}$.

(7) Let us consider real numbers a, b, p, q, s. Suppose $a > 0$ and $p > 0$ and $\frac{s-b}{a} < \frac{s-q}{-p}$. Then $\frac{s-b}{a} < \frac{q-b}{a+p} < \frac{s-q}{-p}$.

2. The Set of Membership Functions

Let X be a non empty set. The functor Membership-Funcs(X) yielding a set is defined by

(Def. 1) for every object f, $f \in it$ iff f is a membership function of X.

Now we state the propositions:

(8) Let us consider a non empty set X, and an object x. Suppose $x \in$ Membership-Funcs(X). Then there exists a membership function f of X such that

   (i) $x = f$, and

   (ii) $\operatorname{dom} f = X$.

(9) Membership-Funcs($\mathbb{R}$) $= \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: f is a fuzzy set of $\mathbb{R}\}$. The theorem is a consequence of (8).

(10) Let us consider non empty sets A, X.
Then $\{\chi_{A,X}\} \subseteq$ Membership-Funcs(X).

(11) $\{\chi_{[a,b],\mathbb{R}}$, where a, b are real numbers : $a \leqslant b\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(12) $\{\chi_{A,\mathbb{R}}$, where A is a subset of $\mathbb{R}$: $A \subseteq \mathbb{R}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(13) $\{f$, where f is a fuzzy set of $\mathbb{R}$: there exists a subset A of $\mathbb{R}$ such that $f = \chi_{A,\mathbb{R}}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(14) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a real number a. Suppose g is continuous and for every real number x, $f(x) = \min(a, g(x))$. Then f is continuous.

PROOF: For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. □

Let us consider functions F, f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(15) If f is continuous and g is continuous and for every real number x, $F(x) = \min(f(x), g(x))$, then F is continuous.

PROOF: For every real number x_0 such that $x_0 \in \operatorname{dom} F$ holds F is continuous in x_0. □

(16) If f is continuous and g is continuous and for every real number x, $F(x) = \max(f(x), g(x))$, then F is continuous.

PROOF: For every real number x_0 such that $x_0 \in \operatorname{dom} F$ holds F is continuous in x_0. □

(17) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and a real number a. Suppose g is continuous and for every real number x, $f(x) = \max(a, g(x))$. Then f is continuous. The theorem is a consequence of (16).

(18) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b. Suppose g is continuous and for every real number x, $f(x) = \max(a, \min(b, g(x)))$. Then f is continuous.

PROOF: Define $\mathcal{H}$(element of $\mathbb{R}$) $= (\min(b, g(\$_1)))(\in \mathbb{R})$. Consider h being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number x, $h(x) = \min(b, g(x))$. h is continuous. For every real number x, $f(x) = \max(a, h(x))$. □

(19) Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Suppose g is continuous and for every real number x, $f(x) = \max(0, \min(1, g(x)))$. Then f is continuous.

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(20) If for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$, then f is continuous.

PROOF: For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. □

(21) If for every real number x, $f(x) = \frac{1}{2} \cdot (\sin(a \cdot x + b)) + \frac{1}{2}$, then f is continuous.

(22) Let us consider real numbers r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(r, \min(s, x))$. Then f is Lipschitzian. The theorem is a consequence of (1).

(23) Let us consider a function g from $\mathbb{R}$ into $\mathbb{R}$. Then $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \min(1, \max(0, g(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $f_0 = f$ and for every real number x, $f(x) = \min(1, \max(0, g(x)))$. $\operatorname{rng} f \subseteq [0, 1]$. □

(24) $\{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Let us consider functions f, g from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(25) If for every real number x, $f(x) = \max(0, \min(1, g(x)))$, then f is a fuzzy set of $\mathbb{R}$.

(26) If for every real number x, $f(x) = \min(1, \max(0, g(x)))$, then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (23).

(27) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. $\operatorname{rng} f \subseteq [0, 1]$. □

(28) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$, a, b being real numbers such that $x = f$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. $\operatorname{rng} f \subseteq [0, 1]$. □

(29) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (28).

(30) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. $\operatorname{rng} f \subseteq [0, 1]$. □

(31) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$.

Suppose for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (30).

(32) Let us consider real numbers a, b, and a fuzzy set f of $\mathbb{R}$. Suppose $a \neq 0$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(33) Let us consider a fuzzy set f of $\mathbb{R}$. Suppose $f \in \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that $a \neq 0$ and for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}\}$. Then f is normalized.

Proof: Consider f_2 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $f = f_2$ and there exist real numbers a, b such that $a \neq 0$ and for every real number t_1, $f_2(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. Consider a, b being real numbers such that $a \neq 0$ and for every real number t_1, $f_2(t_1) = \frac{1}{2} \cdot (\sin(a \cdot t_1 + b)) + \frac{1}{2}$. There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(34) Let us consider a fuzzy set f of $\mathbb{R}$, and real numbers a, b. Suppose $a \neq 0$ and for every real number t_1, $f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}$. Then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. □

(35) Let us consider a fuzzy set f of $\mathbb{R}$. Suppose $f \in \{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that $a \neq 0$ and for every real number $t_1, f(t_1) = \frac{1}{2} \cdot (\cos(a \cdot t_1 + b)) + \frac{1}{2}\}$. Then f is normalized. The theorem is a consequence of (34).

(36) Let us consider a function F from $\mathbb{R}$ into $\mathbb{R}$, real numbers a, b, c, d, and an integer i. Suppose $a \neq 0$ and $i \neq 0$ and for every real number x, $F(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then F is $(\frac{2 \cdot \pi}{a} \cdot i)$-periodic.

Proof: For every real number x such that $x \in \operatorname{dom} F$ holds $x + \frac{2 \cdot \pi}{a} \cdot i$, $x - \frac{2 \cdot \pi}{a} \cdot i \in \operatorname{dom} F$ and $F(x) = F(x + \frac{2 \cdot \pi}{a} \cdot i)$. □

(37) Let us consider a function F from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $F(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then F is periodic.

Proof: There exists a real number t such that F is t-periodic by (36), [6, (1)]. □

(38) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $t_1, f(t_1) = \max(0, \sin(a \cdot t_1 + b))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \max(0, \sin(a \cdot t_1 + b))$. $\operatorname{rng} f \subseteq [0, 1]$ by [5, (4)]. □

(39) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$.

Suppose for every real number x, $f(x) = \max(0, \sin(a \cdot x + b))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (38).

(40) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: there exist real numbers a, b such that for every real number $x, f(x) = \max(0, \cos(a \cdot x + b))\} \subseteq$ Mem– bership-Funcs($\mathbb{R}$).

PROOF: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$ such that $x = f$ and there exist real numbers a, b such that for every real number t_1, $f(t_1) = \max(0, \cos(a \cdot t_1 + b))$. $\operatorname{rng} f \subseteq [0, 1]$. $\square$

(41) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(0, \cos(a \cdot x + b))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (40).

(42) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))\} \subseteq \{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\}$.

(43) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

PROOF: Consider f being a function from $\mathbb{R}$ into $\mathbb{R}$, a, b, c, d being real numbers such that $f = g$ and for every real number x, $f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. f is a fuzzy set of $\mathbb{R}$. $\square$

(44) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (43).

(45) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))\} \subseteq \{f$, where f, g are functions from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \max(0, \min(1, g(x)))\}$.

(46) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(47) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (\arctan(a \cdot x + b)) + d))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (68) and (24).

(48) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d, r, s. Suppose for every real number x, $f(x) = \max(r, \min(s, c \cdot (\sin(a \cdot x + b)) + d))$. Then f is Lipschitzian.

PROOF: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1)-f(x_2)| \leqslant r\cdot|x_1-x_2|$. $\square$

(49) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, d. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (\sin(a \cdot x + b)) + d))$. Then f is Lipschitzian.

Let us consider real numbers a, b and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(50) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is a fuzzy set of $\mathbb{R}$.

Proof: $\operatorname{rng} f \subseteq [0,1]$. $\square$

(51) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is a fuzzy set of $\mathbb{R}$.

Proof: For every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$. $\square$

(52) Let us consider real numbers a, b. Suppose $b \neq 0$. Then $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}$: for every real number $x, f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})\}$ $\subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (51).

Let us consider real numbers a, b and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(53) If for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$. $\square$

(54) If $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$, then f is strictly normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$ and for every element y of $\mathbb{R}$ such that $f(y) = 1$ holds $y = x$ by [11, (20)], (4). $\square$

(55) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b \neq 0$ and for every real number x, $f(x) = \exp(-\frac{(x-a)^2}{2\cdot b^2})$. Then f is continuous.

Proof: Set $h = \text{AffineMap}(1, -a)$. $f = (\text{the function } \exp) \cdot ((\frac{-1}{2\cdot b^2} \cdot h) \cdot h)$. $\square$

(56) Let us consider real numbers a, b, c, r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $b \neq 0$ and for every real number x, $f(x) =$ $\max(r, \min(s, \exp(-\frac{(x-a)^2}{2\cdot b^2}) + c))$. Then f is continuous.

Proof: Define $\mathcal{H}(\text{element of } \mathbb{R}) = (\exp(-\frac{(\$_1-a)^2}{2\cdot b^2}))(\in \mathbb{R})$. Consider h being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element x of $\mathbb{R}$, $h(x) = \mathcal{H}(x)$. For every real number x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0. $\square$

Let us consider real numbers a, b, c and a function f from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(57) Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))$. Then f is continuous.

(58) Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (25).

(59) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c$ are real numbers : $b \neq 0$ and for every real number $x, f(x) = \max(0, \min(1, \exp(-\frac{(x-a)^2}{2 \cdot b^2}) + c))\} \subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (58).

(60) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, r, s. Suppose for every real number x, $f(x) = \max(r, \min(s, (\text{AffineMap}(a, b))(x)))$. Then f is Lipschitzian.

PROOF: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(61) If for every real number x, $f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))$, then f is Lipschitzian.

(62) If for every real number x, $f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))$, then f is a fuzzy set of $\mathbb{R}$.

(63) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : for every real number $x, f(x) = \max(0, \min(1, (\text{AffineMap}(a, b))(x)))\} \subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (25).

(64) Let us consider real numbers a, b, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$. Then f is a fuzzy set of $\mathbb{R}$.

PROOF: $\operatorname{rng} f \subseteq [0, 1]$. $\square$

(65) Let us consider real numbers a, b. Suppose $b > 0$. Let us consider a real number x. Then $(\text{TriangularFS}((a-b), a, (a+b)))(x) = \max(0, 1 - |\frac{x-a}{b}|)$.

PROOF: Set $f_1 = (\text{AffineMap}(0, 0)) \upharpoonright \mathbb{R} \setminus]a-b, a+b[$.
Set $f_2 = (\text{AffineMap}(\frac{1}{a-(a-b)}, -\frac{a-b}{a-(a-b)})) \upharpoonright [a-b, a]$.
Set $f_3 = (\text{AffineMap}(-\frac{1}{a+b-a}, \frac{a+b}{a+b-a})) \upharpoonright [a, a+b]$. Set $F = (f_1 + \cdot f_2) + \cdot f_3$. $F(x) = \max(0, 1 - |\frac{x-a}{b}|)$. $\square$

Let us consider real numbers a, b and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(66) If $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$, then f is triangular. The theorem is a consequence of (65).

(67) If $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$, then f is strictly normalized.

Proof: There exists an element x of $\mathbb{R}$ such that $f(x) = 1$ and for every element y of $\mathbb{R}$ such that $f(y) = 1$ holds $y = x$. □

(68) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose for every real number x, $f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (25).

(69) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b. Suppose $b > 0$ and for every real number x, $f(x) = \max(0, 1 - |\frac{x-a}{b}|)$. Then f is continuous.

Proof: $f = \mathrm{TriangularFS}((a - b), a, (a + b))$. □

(70) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c, r, s. Suppose $b \neq 0$ and for every real number x, $f(x) = \max(r, \min(s, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is Lipschitzian.

Proof: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. □

(71) Let us consider a function f from $\mathbb{R}$ into $\mathbb{R}$, and real numbers a, b, c. Suppose $b \neq 0$ and for every real number x, $f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))$. Then f is Lipschitzian.

(72) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : $b > 0$ and for every real number $x, f(x) = \max(0, 1 - |\frac{x-a}{b}|)\} \subseteq$ Membership-Funcs($\mathbb{R}$).

Proof: $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b$ are real numbers : $b > 0$ and for every real number $x, f(x) = \max(0, 1 - |\frac{x-a}{b}|)\} \subseteq \{\mathrm{TriangularFS}(a, b, c)$, where a, b, c are real numbers : $a < b < c\}$. □

(73) $\{f$, where f is a function from $\mathbb{R}$ into $\mathbb{R}, a, b, c, d$ are real numbers : $b \neq 0$ and for every real number $x, f(x) = \max(0, \min(1, c \cdot (1 - |\frac{x-a}{b}|)))\} \subseteq$ Membership-Funcs($\mathbb{R}$). The theorem is a consequence of (68).

(74) Let us consider real numbers a, b, p, q, s. Then $(\mathrm{AffineMap}(a, b)) {\restriction}]-\infty, s[+\cdot (\mathrm{AffineMap}(p, q)) {\restriction} [s, +\infty[$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(75) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a, b)) {\restriction}]-\infty, \frac{q-b}{a-p}[+\cdot (\mathrm{AffineMap}(p, q)) {\restriction} [\frac{q-b}{a-p}, +\infty[)(x)))$. Then f is a fuzzy set of $\mathbb{R}$. The theorem is a consequence of (74) and (25).

(76) Let us consider real numbers a, b, c. Suppose $a < b < c$. Then

(i) $(\text{TriangularFS}(a, b, c))(a) = 0$, and

(ii) $(\text{TriangularFS}(a, b, c))(b) = 1$, and

(iii) $(\text{TriangularFS}(a, b, c))(c) = 0$.

(77) Let us consider real numbers a, b, c, d. Suppose $a < b < c < d$. Then

(i) $(\text{TrapezoidalFS}(a, b, c, d))(a) = 0$, and

(ii) $(\text{TrapezoidalFS}(a, b, c, d))(b) = 1$, and

(iii) $(\text{TrapezoidalFS}(a, b, c, d))(c) = 1$, and

(iv) $(\text{TrapezoidalFS}(a, b, c, d))(d) = 0$.

Let us consider real numbers a, b, p, q and a real number x. Now we state the propositions:

(78) Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$ and $\frac{1-b}{a} = \frac{1-q}{-p}$. Then $(\text{TriangularFS}(\frac{-b}{a}, \frac{1-b}{a}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$.
PROOF: For every real number x, $(\text{TriangularFS}(\frac{-b}{a}, \frac{1-b}{a}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$. $\square$

(79) Suppose $a > 0$ and $p > 0$ and $\frac{1-b}{a} < \frac{1-q}{-p}$.
Then $(\text{TrapezoidalFS}(\frac{-b}{a}, \frac{1-b}{a}, \frac{1-q}{-p}, \frac{q}{p}))(x) = \max(0, \min(1, ((\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$.
PROOF: Set $f_4 = (\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[$.
Set $f_5 = (\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[$.
For every real number x, $(\text{TrapezoidalFS}(\frac{-b}{a}, \frac{1-b}{a}, \frac{1-q}{-p}, \frac{q}{p}))(x) = \max(0, \min(1, (f_4 +\cdot f_5)(x)))$. $\square$

(80) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and $f = (\text{AffineMap}(a, b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\text{AffineMap}(-p, q))\restriction[\frac{q-b}{a+p}, +\infty[$. Then f is Lipschitzian.
PROOF: There exists a real number r such that $0 < r$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f(x_1) - f(x_2)| \leqslant r \cdot |x_1 - x_2|$. $\square$

(81) Let us consider real numbers a, b, p, q. Suppose $a > 0$ and $p > 0$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \mathrm{dom}((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)$ holds $|((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)(x_1)-((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)(x_2)| \leqslant r \cdot |x_1 - x_2|$.

The theorem is a consequence of (74) and (80).

(82) Let us consider real numbers a, b, p, q, r, s, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and for every real number x, $f(x) = \max(r, \min(s, ((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is Lipschitzian. The theorem is a consequence of (74), (81), and (1).

(83) Let us consider real numbers a, b, c. Suppose $a < b < c$. Let us consider a real number x. Then $(\mathrm{TriangularFS}(a,b,c))(x) = \max(0, \min(1, ((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction]-\infty, b[+\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b, +\infty[)(x)))$. The theorem is a consequence of (78).

(84) Let us consider real numbers a, b, c, d. Suppose $a < b < c < d$. Let us consider a real number x. Then $(\mathrm{TrapezoidalFS}(a,b,c,d))(x) = \max(0, \min(1, ((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction]-\infty, \frac{b \cdot d - a \cdot c}{d-c+b-a}[+\cdot(\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c}))\restriction[\frac{b \cdot d - a \cdot c}{d-c+b-a}, +\infty[)(x)))$. The theorem is a consequence of (79).

(85) Let us consider real numbers a, b, p, q, and a function f from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a > 0$ and $p > 0$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is Lipschitzian.

(86) Let us consider real numbers a, b, c. If $a < b < c$, then $\mathrm{TriangularFS}(a,b,c)$ is Lipschitzian. The theorem is a consequence of (83) and (82).

(87) Let us consider real numbers a, b, c, d. If $a < b < c < d$, then $\mathrm{TrapezoidalFS}(a,b,c,d)$ is Lipschitzian. The theorem is a consequence of (84) and (82).

Let us consider real numbers a, b, p, q and a fuzzy set f of $\mathbb{R}$. Now we state the propositions:

(88) Suppose $a > 0$ and $p > 0$ and $\frac{-b}{a} < \frac{q}{p}$ and $\frac{1-b}{a} = \frac{1-q}{-p}$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}(-p,q))\restriction[\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is triangular and strictly normalized. The theorem is a consequence of (78).

(89) Suppose $a > 0$ and $p > 0$ and $\frac{1-b}{a} < \frac{1-q}{-p}$ and for every real number x, $f(x) = \max(0, \min(1, ((\mathrm{AffineMap}(a,b))\restriction]-\infty, \frac{q-b}{a+p}[+\cdot(\mathrm{AffineMap}$

$(-p, q)) \restriction [\frac{q-b}{a+p}, +\infty[)(x)))$. Then f is trapezoidal and normalized. The theorem is a consequence of (79).

(90) $\{f$, where f is a fuzzy set of $\mathbb{R}$: f is triangular$\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(91) $\{$TriangularFS(a, b, c), where a, b, c are real numbers : $a < b < c\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(92) $\{f$, where f is a fuzzy set of $\mathbb{R}$: f is trapezoidal$\} \subseteq$ Membership-Funcs($\mathbb{R}$).

(93) $\{$TrapezoidalFS(a, b, c, d), where a, b, c, d are real numbers : $a < b < c < d\} \subseteq$ Membership-Funcs($\mathbb{R}$).

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