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Algorithm NextFit for the Bin Packing Problem[1]

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Summary. The bin packing problem is a fundamental and important optimization problem in theoretical computer science [4], [6]. An instance is a sequence of items, each being of positive size at most one. The task is to place all the items into bins so that the total size of items in each bin is at most one and the number of bins that contain at least one item is minimum.

Approximation algorithms have been intensively studied. Algorithm NextFit would be the simplest one. The algorithm repeatedly does the following: If the first unprocessed item in the sequence can be placed, in terms of size, additionally to the bin into which the algorithm has placed an item the last time, place the item into that bin; otherwise place the item into an empty bin. Johnson [5] proved that the number of the resulting bins by algorithm NextFit is less than twice the number of the fewest bins that are needed to contain all items.

In this article, we formalize in Mizar [1], [2] the bin packing problem as follows: An instance is a sequence of positive real numbers that are each at most one. The task is to find a function that maps the indices of the sequence to positive integers such that the sum of the subsequence for each of the inverse images is at most one and the size of the image is minimum. We then formalize algorithm NextFit, its feasibility, its approximation guarantee, and the tightness of the approximation guarantee.

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1. Preliminaries

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and i be an element of $\operatorname{dom} a$. Let us observe that the functor $a(i)$ yields an element of $\mathbb{R}$. Let h be a non empty finite sequence of elements of $\mathbb{N}^*$ and i be an element of $\operatorname{dom} h$. Let us observe that the functor $h(i)$ yields a finite sequence of elements of $\mathbb{N}$. Now we state the propositions:

(1) Let us consider a natural number n. If n is odd, then $1 \leqslant n$ and $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$.

(2) Let us consider a set D, and a finite sequence p. Suppose for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in D$. Then p is a finite sequence of elements of D.

(3) Let us consider objects x, y. Then $\{\langle x, y\rangle\}^{-1}(\{y\}) = \{x\}$.
PROOF: For every object v, $v \in \{x\}$ iff $v \in \operatorname{dom}\{\langle x, y\rangle\}$ and $\{\langle x, y\rangle\}(v) \in \{y\}$. $\square$

(4) Let us consider natural numbers a, b, and a set s. If $\operatorname{Seg} a \cup \{s\} = \operatorname{Seg} b$, then $a = b$ or $a + 1 = b$. PROOF: $b - a \leqslant 1$. $\square$

Let D be a non empty set, f be a D-valued finite sequence, and I be a set. The functor $\operatorname{Seq}(f, I)$ yielding a D-valued finite sequence is defined by the term

(Def. 1) $\operatorname{Seq}(f \restriction I)$.

Let a be a non empty finite sequence of elements of $\mathbb{R}$, f be a function, and s be a set. The functor $\operatorname{SumBin}(a, f, s)$ yielding a real number is defined by the term

(Def. 2) $\sum \operatorname{Seq}(a, f^{-1}(s))$.

Let us observe that there exists a non empty finite sequence of elements of $\mathbb{R}$ which is positive. Let a be a finite sequence of elements of $\mathbb{R}$. We say that a is at most one if and only if

(Def. 3) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $a(i) \leqslant 1$.

Note that there exists a non empty, positive finite sequence of elements of $\mathbb{R}$ which is at most one. Let us consider a finite sequence f of elements of $\mathbb{N}$ and natural numbers j, b. Now we state the propositions:

(5) If $b = j$, then $(f \frown \langle b\rangle)^{-1}(\{j\}) = f^{-1}(\{j\}) \cup \{\operatorname{len} f + 1\}$.
PROOF: For every object z, $z \in (f \frown \langle b\rangle)^{-1}(\{j\})$ iff $z \in f^{-1}(\{j\}) \cup \{\operatorname{len} f + 1\}$. $\square$

(6) If $b \neq j$, then $(f \frown \langle b\rangle)^{-1}(\{j\}) = f^{-1}(\{j\})$.
PROOF: For every object z, $z \in (f \frown \langle b\rangle)^{-1}(\{j\})$ iff $z \in f^{-1}(\{j\})$. $\square$

(7) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a set p, and a natural number i. Suppose $p \cup \{i\} \subseteq \operatorname{dom} a$ and for every natural

number m such that $m \in p$ holds $m < i$. Then $\mathrm{Seq}(a \restriction (p \cup \{i\})) = \mathrm{Seq}(a \restriction p) \frown \langle a(i) \rangle$.

Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and natural numbers j, b. Now we state the propositions:

(8) Suppose $\mathrm{len}\, f + 1 \leqslant \mathrm{len}\, a$. Then if $b = j$, then $\mathrm{SumBin}(a, f \frown \langle b \rangle, \{j\}) = \mathrm{SumBin}(a, f, \{j\}) + a(\mathrm{len}\, f + 1)$.
PROOF: $(f \frown \langle b \rangle)^{-1}(\{j\}) = f^{-1}(\{j\}) \cup \{\mathrm{len}\, f + 1\}$. For every natural number m such that $m \in f^{-1}(\{j\})$ holds $m < \mathrm{len}\, f + 1$. $\square$

(9) Suppose $\mathrm{len}\, f + 1 \leqslant \mathrm{len}\, a$. Then if $b \neq j$, then $\mathrm{SumBin}(a, f \frown \langle b \rangle, \{j\}) = \mathrm{SumBin}(a, f, \{j\})$.

(10) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\mathrm{dom}\, f = \mathrm{dom}\, a$. Then $\mathrm{SumBin}(a, f, \mathrm{rng}\, f) = \sum a$.

(11) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and sets s, t. Suppose $\mathrm{dom}\, f \subseteq \mathrm{dom}\, a$ and s misses t. Then $\mathrm{SumBin}(a, f, s \cup t) = \mathrm{SumBin}(a, f, s) + \mathrm{SumBin}(a, f, t)$.
PROOF: Reconsider $F = a$ as a partial function from $\mathbb{N}$ to $\mathbb{R}$. For every set W such that $W \subseteq \mathrm{dom}\, a$ holds $\sum_{\kappa=0}^{W} F(\kappa) = \sum \mathrm{Seq}(a, W)$ by [3, (51)]. $\square$

(12) Let us consider a non empty, positive finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and a set s. If $\mathrm{dom}\, f \subseteq \mathrm{dom}\, a$, then $0 \leqslant \mathrm{SumBin}(a, f, s)$.
PROOF: Reconsider $s_1 = \mathrm{Seq}(a, f^{-1}(s))$ as a real-valued finite sequence. For every natural number i such that $i \in \mathrm{dom}\, s_1$ holds $0 \leqslant s_1(i)$. $\square$

(13) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, and a set s. If s misses $\mathrm{rng}\, f$, then $\mathrm{SumBin}(a, f, s) = 0$.

2. OPTIMAL PACKING

Now we state the propositions:

(14) Let us consider a non empty, at most one finite sequence a of elements of $\mathbb{R}$. Then there exists a natural number k and there exists a non empty finite sequence f of elements of $\mathbb{N}$ such that $\mathrm{dom}\, f = \mathrm{dom}\, a$ and for every natural number j such that $j \in \mathrm{rng}\, f$ holds $\mathrm{SumBin}(a, f, \{j\}) \leqslant 1$ and $k = \overline{\overline{\mathrm{rng}\, f}}$.
PROOF: Set $k_1 = \mathrm{len}\, a$. Set $f_1 = \mathrm{idseq}(k_1)$. For every natural number j such that $j \in \mathrm{rng}\, f_1$ holds $\mathrm{SumBin}(a, f_1, \{j\}) \leqslant 1$. There exists a non

empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ and $k_1 = \overline{\overline{\operatorname{rng} f}}$. □

(15) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$. Then there exists a finite sequence f_2 of elements of $\mathbb{N}$ such that

(i) $\operatorname{dom} f_2 = \operatorname{dom} a$, and

(ii) for every natural number j such that $j \in \operatorname{rng} f_2$ holds $\operatorname{SumBin}(a, f_2, \{j\}) \leqslant 1$, and

(iii) there exists a natural number k such that $\operatorname{rng} f_2 = \operatorname{Seg} k$, and

(iv) $\overline{\overline{\operatorname{rng} f}} = \overline{\overline{\operatorname{rng} f_2}}$.

Proof: Reconsider $g_3 = \operatorname{Sgm}_0 \operatorname{rng} f$ as a finite 0-sequence of $\mathbb{N}$. Reconsider $g_2 = \operatorname{XFS2FS}(g_3)$ as a one-to-one function. Reconsider $g = {g_2}^{-1}$ as a one-to-one function. Reconsider $f_3 = g \cdot f$ as a finite sequence. Consider k_0 being a natural number such that $\operatorname{dom} g_2 = \operatorname{Seg} k_0$. For every natural number j such that $j \in \operatorname{rng} f_3$ holds $\operatorname{SumBin}(a, f_3, \{j\}) \leqslant 1$. □

Let a be a non empty, at most one finite sequence of elements of $\mathbb{R}$. The functor $\operatorname{Opt}(a)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 4) there exists a non empty finite sequence g of elements of $\mathbb{N}$ such that $\operatorname{dom} g = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} g$ holds $\operatorname{SumBin}(a, g, \{j\}) \leqslant 1$ and $\mathit{it} = \overline{\overline{\operatorname{rng} g}}$ and for every non empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ holds $\mathit{it} \leqslant \overline{\overline{\operatorname{rng} f}}$.

Now we state the propositions:

(16) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a finite sequence f of elements of $\mathbb{N}$, a natural number k, and a real-valued finite sequence R_1. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and $\operatorname{rng} f = \operatorname{Seg} k$ and $\operatorname{len} R_1 = k$ and for every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) = \operatorname{SumBin}(a, f, \{j\})$. Then $\sum R_1 = \operatorname{SumBin}(a, f, \operatorname{rng} f)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real-valued finite sequence r_1 such that $r_1 = R_1 \restriction \operatorname{Seg} \$_1$ holds $\sum r_1 = \operatorname{SumBin}(a, f, \operatorname{Seg} \$_1)$. For every real-valued finite sequence r_1 such that $r_1 = R_1 \restriction \operatorname{Seg} 1$ holds $\sum r_1 = \operatorname{SumBin}(a, f, \operatorname{Seg} 1)$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < k$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant k$ holds $\mathcal{P}[i]$. □

(17) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$. Then $\lceil \sum a \rceil \leqslant \overline{\overline{\operatorname{rng} f}}$.

PROOF: Consider f_2 being a finite sequence of elements of $\mathbb{N}$ such that $\operatorname{dom} f_2 = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f_2$ holds $\operatorname{SumBin}(a, f_2, \{j\}) \leqslant 1$ and there exists a natural number k such that $\operatorname{rng} f_2 = \operatorname{Seg} k$ and $\overline{\overline{\operatorname{rng} f}} = \overline{\overline{\operatorname{rng} f_2}}$. Consider i being a natural number such that $\operatorname{rng} f_2 = \operatorname{Seg} i$. Define $\mathcal{N}(\text{natural number}) = \operatorname{SumBin}(a, f_2, \{\$_1\})$.

There exists a finite sequence p such that $\operatorname{len} p = i$ and for every natural number j such that $j \in \operatorname{dom} p$ holds $p(j) = \mathcal{N}(j)$. Consider R_1 being a finite sequence such that $\operatorname{len} R_1 = i$ and for every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) = \operatorname{SumBin}(a, f_2, \{j\})$. For every natural number j such that $j \in \operatorname{dom} R_1$ holds $R_1(j) \in \mathbb{R}$. R_1 is a finite sequence of elements of $\mathbb{R}$.

Reconsider $R_2 = i \mapsto 1$ as a real-valued, i-element finite sequence. For every natural number j such that $j \in \operatorname{Seg} i$ holds $R_1(j) \leqslant R_2(j)$. $\sum R_1 = \operatorname{SumBin}(a, f_2, \operatorname{rng} f_2)$. $\sum a \leqslant \overline{\overline{\operatorname{rng} f}}$. □

(18) Let us consider a non empty, at most one finite sequence a of elements of $\mathbb{R}$. Then $\lceil \sum a \rceil \leqslant \operatorname{Opt}(a)$. The theorem is a consequence of (17).

3. Online Algorithms

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and A be a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. The functor OnlinePackingHistory(a, A) yielding a non empty finite sequence of elements of $\mathbb{N}^*$ is defined by

(Def. 5) $\operatorname{len} it = \operatorname{len} a$ and $it(1) = \langle 1 \rangle$ and for every natural number i such that $1 \leqslant i < \operatorname{len} a$ there exists an element d_1 of $\mathbb{R}$ and there exists a finite sequence d_2 of elements of $\mathbb{N}$ such that $d_1 = a(i+1)$ and $d_2 = it(i)$ and $it(i+1) = d_2 \frown \langle A(d_1, d_2) \rangle$.

Now we state the propositions:

(19) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. Then $(\text{OnlinePackingHistory}(a, A))(1) = \{\langle 1, 1 \rangle\}$.

(20) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \text{OnlinePackingHistory}(a, A)$.
Then $\operatorname{SumBin}(a, h(1), \{h(1)(1)\}) = a(1)$. The theorem is a consequence of (3).

Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a natural number i. Now we state the propositions:

(21) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i \leqslant \mathrm{len}\, a$, then $h(i)$ is a finite sequence of elements of $\mathbb{N}$.

(22) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i \leqslant \mathrm{len}\, a$, then $\mathrm{len}\, h(i) = i$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv \mathrm{len}\, h(\$_1) = \$_1$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \mathrm{len}\, a$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{P}[i]$. For every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{P}[i]$. $\square$

(23) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i < \mathrm{len}\, a$, then $h(i+1) = h(i) \frown \langle A(a(i+1), h(i)) \rangle$ and $h(i+1)(i+1) = A(a(i+1), h(i))$. The theorem is a consequence of (22).

(24) If $h = \mathrm{OnlinePackingHistory}(a, A)$, then if $1 \leqslant i < \mathrm{len}\, a$, then $\mathrm{rng}\, h(i+1) = \mathrm{rng}\, h(i) \cup \{h(i+1)(i+1)\}$. The theorem is a consequence of (23).

(25) Let us consider a non empty, positive finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, A)$. Let us consider natural numbers i, l. Suppose $1 \leqslant i < \mathrm{len}\, a$. Then $\mathrm{SumBin}(a, h(i), \{l\}) \leqslant \mathrm{SumBin}(a, h(i+1), \{l\})$. The theorem is a consequence of (21), (22), (23), (8), and (6).

Let a be a non empty finite sequence of elements of $\mathbb{R}$ and A be a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$. The functor $\mathrm{OnlinePacking}(a, A)$ yielding a non empty finite sequence of elements of $\mathbb{N}$ is defined by the term

(Def. 6) $(\mathrm{OnlinePackingHistory}(a, A))(\mathrm{len}\, \mathrm{OnlinePackingHistory}(a, A))$.

Now we state the proposition:

(26) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, a function A from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a non empty finite sequence f of elements of $\mathbb{N}$. Then $\mathrm{dom}(\mathrm{OnlinePacking}(a, A)) = \mathrm{dom}\, a$. The theorem is a consequence of (22).

4. Feasibility of Algorithm NextFit

Let a be a non empty finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{NextFit}(a)$ yielding a function from $\mathbb{R} \times \mathbb{N}^*$ into $\mathbb{N}$ is defined by

(Def. 7) for every real number s and for every finite sequence f of elements of $\mathbb{N}$, if $s + \mathrm{SumBin}(a, f, \{f(\mathrm{len}\, f)\}) \leqslant 1$, then $\mathit{it}(s, f) = f(\mathrm{len}\, f)$ and if $s + \mathrm{SumBin}(a, f, \{f(\mathrm{len}\, f)\}) > 1$, then $\mathit{it}(s, f) = f(\mathrm{len}\, f) + 1$.

Now we state the propositions:

(27) Let us consider a non empty finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$.
Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider a natural number i. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$. Then there exists a natural number k such that

(i) $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$, and

(ii) $h(i)(i) = k$.

PROOF: Define $\mathcal{R}[\text{natural number}] \equiv$ there exists a natural number k such that $\mathrm{rng}\, h(\$_1) = \mathrm{Seg}\, k$ and $h(\$_1)(\$_1) = k$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \mathrm{len}\, a$ and $\mathcal{R}[i]$ holds $\mathcal{R}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{R}[i]$. For every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{R}[i]$. $\square$

(28) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider a natural number i. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$. Then $\mathrm{SumBin}(a, h(i), \{h(i)(i)\}) \leqslant 1$.
PROOF: Define $\mathcal{T}[\text{natural number}] \equiv \mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) \leqslant 1$. $\mathrm{SumBin}(a, h(1), \{h(1)(1)\}) \leqslant 1$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \mathrm{len}\, a$ and $\mathcal{T}[i]$ holds $\mathcal{T}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{T}[i]$. For every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{T}[i]$. $\square$

(29) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, j. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$ and $j \in \mathrm{rng}\, h(i)$. Then $\mathrm{SumBin}(a, h(i), \{j\}) \leqslant 1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every natural number j such that $j \in \mathrm{rng}\, h(\$_1)$ holds $\mathrm{SumBin}(a, h(\$_1), \{j\}) \leqslant 1$. For every natural number j such that $j \in \mathrm{rng}\, h(1)$ holds $\mathrm{SumBin}(a, h(1), \{j\}) \leqslant 1$. For every element i_0 of $\mathbb{N}$ such that $1 \leqslant i_0 < \mathrm{len}\, a$ and $\mathcal{P}[i_0]$ holds $\mathcal{P}[i_0 + 1]$.

For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{P}[i]$. For every natural numbers i, j such that $1 \leqslant i \leqslant \mathrm{len}\, a$ and $j \in \mathrm{rng}\, h(i)$ holds $\mathrm{SumBin}(a, h(i), \{j\}) \leqslant 1$. $\square$

(30) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Let us consider a natural number j. If $j \in \mathrm{rng}\, f$, then $\mathrm{SumBin}(a, f, \{j\}) \leqslant 1$. The theorem is a consequence of (29).

5. Approximation Guarantee of Algorithm NextFit

Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and natural numbers i, k. Now we state the propositions:

(31) If $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$, then if $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$, then $h(i)(i) = k$. The theorem is a consequence of (27).

(32) Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Then suppose $1 \leqslant i < \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $\mathrm{rng}\, h(i+1) = \mathrm{Seg}(k+1)$. Then $\mathrm{SumBin}(a, h(i+1), \{k\}) + \mathrm{SumBin}(a, h(i+1), \{k+1\}) > 1$. The theorem is a consequence of (21), (22), (23), (31), (24), (6), (8), and (12).

(33) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, l, k. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$. Then $\mathrm{SumBin}(a, h(i), \{l\}) + \mathrm{SumBin}(a, h(i), \{l+1\}) > 1$.
Proof: Define $\mathcal{N}[\text{natural number}] \equiv$ for every natural number l for every natural number k such that $\mathrm{rng}\, h(\$_1) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(\$_1), \{l\}) + \mathrm{SumBin}(a, h(\$_1), \{l+1\}) > 1$. For every natural number l and for every natural number k such that $\mathrm{rng}\, h(1) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(1), \{l\}) + \mathrm{SumBin}(a, h(1), \{l+1\}) > 1$.

For every element i_0 of $\mathbb{N}$ such that $1 \leqslant i_0 < \mathrm{len}\, a$ and $\mathcal{N}[i_0]$ holds $\mathcal{N}[i_0 + 1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{N}[i]$. For every natural numbers i, l, k such that $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant l < k$ holds $\mathrm{SumBin}(a, h(i), \{l\}) + \mathrm{SumBin}(a, h(i), \{l+1\}) > 1$. □

(34) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence h of elements of $\mathbb{N}^*$. Suppose $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Let us consider natural numbers i, j, k. Suppose $1 \leqslant i \leqslant \mathrm{len}\, a$ and $\mathrm{rng}\, h(i) = \mathrm{Seg}\, k$ and $2 \leqslant k$ and $1 \leqslant j \leqslant k \operatorname{div} 2$. Then $\mathrm{SumBin}(a, h(i), \{2 \cdot j - 1\}) + \mathrm{SumBin}(a, h(i), \{2 \cdot j\}) > 1$. The theorem is a consequence of (33).

(35) Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence h of elements of $\mathbb{N}^*$, and a finite sequence f of elements of $\mathbb{N}$. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Then there exists a natural number k such that $\mathrm{rng}\, f = \mathrm{Seg}\, k$. The theorem is a consequence of (27).

(36) Let us consider a non empty, positive, at most one finite sequence a of

elements of $\mathbb{R}$, a non empty finite sequence f of elements of $\mathbb{N}$, and a natural number k. Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $\mathrm{rng}\, f = \mathrm{Seg}\, k$. Let us consider a natural number j. Suppose $1 \leqslant j \leqslant k \,\mathrm{div}\, 2$. Then $\mathrm{SumBin}(a, f, \{2 \cdot j - 1\}) + \mathrm{SumBin}(a, f, \{2 \cdot j\}) > 1$. The theorem is a consequence of (34).

Let us consider a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, a non empty finite sequence f of elements of $\mathbb{N}$, and a natural number k. Now we state the propositions:

(37) If $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$, then $k \,\mathrm{div}\, 2 < \sum a$. The theorem is a consequence of (35), (26), (2), (36), (12), (16), and (10).

(38) Suppose $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$. Then $k \leqslant 2 \cdot \lceil \sum a \rceil - 1$.
PROOF: $k \,\mathrm{div}\, 2 < \lceil \sum a \rceil$. $\frac{k-1}{2} \leqslant k \,\mathrm{div}\, 2$ by [8, (4), (5)]. $\square$

(39) If $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$ and $k = \overline{\overline{\mathrm{rng}\, f}}$, then $k \leqslant 2 \cdot (\mathrm{Opt}(a)) - 1$. The theorem is a consequence of (38) and (18).

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Now we state the propositions:

(40) Let us consider a natural number n, a real number ε, a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose n is odd and $\mathrm{len}\, a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$ and $f = \mathrm{OnlinePacking}(a, \mathrm{NextFit}(a))$. Then $n = \overline{\overline{\mathrm{rng}\, f}}$.
PROOF: $1 \leqslant n$. Set $h = \mathrm{OnlinePackingHistory}(a, \mathrm{NextFit}(a))$. Define $\mathcal{N}[\text{natural number}] \equiv$ if $\$_1$ is odd, then $\mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) = 2 \cdot \varepsilon$ and if $\$_1$ is even, then $\mathrm{SumBin}(a, h(\$_1), \{h(\$_1)(\$_1)\}) = 1 - \varepsilon$ and $h(\$_1)(\$_1) = \$_1$ and $\mathrm{rng}\, h(\$_1) = \mathrm{Seg}\, \$_1$. $\mathcal{N}[1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \mathrm{len}\, a$ and $\mathcal{N}[i]$ holds $\mathcal{N}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \mathrm{len}\, a$ holds $\mathcal{N}[i]$. $\square$

(41) Let us consider a natural number n, a real number ε, and a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$. Suppose n is odd and $\mathrm{len}\, a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$. Then $\sum a = \frac{n+1}{2} + \frac{1}{n+1} - \frac{1}{2}$.
PROOF: $1 \leqslant n$. $n + 1 \,\mathrm{div}\, 2 = \frac{n+1}{2}$. Define $\mathcal{N}[\text{natural number}] \equiv$ if $\$_1$ is odd, then $\sum(a \restriction \$_1) = 2 \cdot \varepsilon \cdot (\$_1 + 1 \,\mathrm{div}\, 2) + (1 - \varepsilon) \cdot ((\$_1 + 1 \,\mathrm{div}\, 2) - 1)$ and

if $\$_1$ is even, then $\sum(a{\restriction}\$_1) = 2 \cdot \varepsilon \cdot (\$_1 \operatorname{div} 2) + (1-\varepsilon) \cdot (\$_1 \operatorname{div} 2)$. For every element i of $\mathbb{N}$ such that $1 \leqslant i < \operatorname{len} a$ and $\mathcal{N}[i]$ holds $\mathcal{N}[i+1]$. For every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} a$ holds $\mathcal{N}[i]$. □

(42) Let us consider a natural number n, a real number ε, a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$, and a non empty finite sequence f of elements of $\mathbb{N}$. Suppose n is odd and $\operatorname{len} a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$ and $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $f(i) = 1$ and if i is even, then $f(i) = (i \operatorname{div} 2) + 1$. Let us consider a natural number j. If $j \in \operatorname{rng} f$, then $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$.

Proof: $1 \leqslant n$. $n + 1 \operatorname{div} 2 = \frac{n+1}{2}$. Set $n_1 = n + 1 \operatorname{div} 2$. $1 + 1 \leqslant n + 1$. For every object y, $y \in \operatorname{Seg} n_1$ iff there exists an object x such that $x \in \operatorname{dom} f$ and $y = f(x)$. □

(43) Let us consider a natural number n, a real number ε, and a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$. Suppose n is odd and $\operatorname{len} a = n$ and $\varepsilon = \frac{1}{n+1}$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds if i is odd, then $a(i) = 2 \cdot \varepsilon$ and if i is even, then $a(i) = 1 - \varepsilon$. Then $n = 2 \cdot (\operatorname{Opt}(a)) - 1$.

Proof: $1 \leqslant n$. $n+1 \operatorname{div} 2 = \frac{n+1}{2}$. There exists a non empty finite sequence g of elements of $\mathbb{N}$ such that $\operatorname{dom} g = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} g$ holds $\operatorname{SumBin}(a, g, \{j\}) \leqslant 1$ and $n + 1 \operatorname{div} 2 = \overline{\overline{\operatorname{rng} g}}$ and for every non empty finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{dom} f = \operatorname{dom} a$ and for every natural number j such that $j \in \operatorname{rng} f$ holds $\operatorname{SumBin}(a, f, \{j\}) \leqslant 1$ holds $n + 1 \operatorname{div} 2 \leqslant \overline{\overline{\operatorname{rng} f}}$. □

(44) Let us consider a natural number n. Suppose n is odd. Then there exists a non empty, positive, at most one finite sequence a of elements of $\mathbb{R}$ such that

(i) $\operatorname{len} a = n$, and

(ii) for every non empty finite sequence f of elements of $\mathbb{N}$ such that $f = \operatorname{OnlinePacking}(a, \operatorname{NextFit}(a))$ holds $n = \overline{\overline{\operatorname{rng} f}}$ and $n = 2 \cdot (\operatorname{Opt}(a)) - 1$.

Proof: $1 \leqslant n$. Set $\varepsilon = \frac{1}{n+1}$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ if $\$_1$ is odd, then $\$_2 = 2 \cdot \varepsilon$ and if $\$_1$ is even, then $\$_2 = 1 - \varepsilon$. For every natural number i such that $i \in \operatorname{Seg} n$ there exists an object x such that $\mathcal{P}[i, x]$. Consider a_0 being a finite sequence such that $\operatorname{dom} a_0 = \operatorname{Seg} n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $\mathcal{P}[i, a_0(i)]$. For every natural number i such that $i \in \operatorname{dom} a_0$ holds $a_0(i) \in \mathbb{R}$. a_0 is positive by (1), [7, (22)]. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} a_0$ holds $a_0(i) \leqslant 1$. $\square$

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