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Ascoli-Arzelà Theorem[1]

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Summary. In this article we formalize the Ascoli-Arzelà theorem [5], [6], [8] in Mizar [1], [2]. First, we gave definitions of equicontinuousness and equiboundedness of a set of continuous functions [12], [7], [3], [9]. Next, we formalized the Ascoli-Arzelà theorem using those definitions, and proved this theorem.

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1. Equicontinuousness and Equiboundedness of Continuous Functions

From now on S, T denote real normed spaces and F denotes a subset of (the carrier of T)$^{(\text{the carrier of } S)}$.

Let X be a non empty metric space and Y be a subset of X. The functor $\overline{Y}$ yielding a subset of X is defined by

(Def. 1) there exists a subset Z of $X_{\rm top}$ such that $Z = Y$ and $it = \overline{Z}$.

Now we state the proposition:

(1) Let us consider a real normed space X, a subset Y of X, and a subset Z of MetricSpaceNorm X. If $Y = Z$, then $\overline{Y} = \overline{Z}$.

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Let X be a non empty metric space and H be a non empty subset of X. Observe that $\overline{H}$ is non empty.

Now we state the propositions:

(2) Let us consider a topological space S, and a finite sequence F of elements of 2^{α}. Suppose for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} F$ holds $F_{/i}$ is compact. Then $\bigcup \operatorname{rng} F$ is compact, where α is the carrier of S.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of $2^{(\text{the carrier of } S)}$ such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} F$ holds $F_{/i}$ is compact holds $\bigcup \operatorname{rng} F$ is compact. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(3) Let us consider a non empty topological space S, a normed linear topological space T, a function f from S into T, and a point x of S. Then f is continuous at x if and only if for every real number e such that $0 < e$ there exists a subset H of S such that H is open and $x \in H$ and for every point y of S such that $y \in H$ holds $\|f(x) - f(y)\| < e$.
PROOF: For every subset G of T such that G is open and $f(x) \in G$ there exists a subset H of S such that H is open and $x \in H$ and $f^{\circ}H \subseteq G$. $\square$

(4) Let us consider a non empty metric space S, a non empty, compact topological space V, a normed linear topological space T, and a function f from V into T. Suppose $V = S_{\text{top}}$. Then f is continuous if and only if for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f_{/x_1} - f_{/x_2}\| < e$.
PROOF: For every point x of V, f is continuous at x. $\square$

Let S be a non empty metric space, T be a real normed space, and F be a subset of $(\text{the carrier of } T)^{(\text{the carrier of } S)}$. We say that F is equibounded if and only if

(Def. 2) there exists a real number K such that for every function f from the carrier of S into the carrier of T such that $f \in F$ for every element x of S, $\|f(x)\| \leqslant K$.

Let x_0 be a point of S. We say that F is equicontinuous at x_0 if and only if

(Def. 3) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, x_0) < d$ holds $\|f(x) - f(x_0)\| < e$.

We say that F is equicontinuous if and only if

(Def. 4) for every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$.

Now we state the proposition:

(5) Let us consider a non empty metric space S, a real normed space T, and a subset F of (the carrier of $T)^{\alpha}$. Suppose S_{top} is compact. Then F is equicontinuous if and only if for every point x of S, F is equicontinuous at x, where α is the carrier of S.

PROOF: Define $\mathcal{P}[\text{element of } S, \text{real number}] \equiv 0 < \$_2$ and for every function f from the carrier of S into the carrier of T such that $f \in F$ for every point x of S such that $\rho(x, \$_1) < \$_2$ holds $\|f(x) - f(\$_1)\| < \frac{e}{2}$. For every element x_0 of the carrier of S, there exists an element d of $\mathbb{R}$ such that $\mathcal{P}[x_0, d]$.

Consider D being a function from the carrier of S into $\mathbb{R}$ such that for every element x_0 of the carrier of S, $\mathcal{P}[x_0, D(x_0)]$. Set $C_1 =$ the set of all $\mathrm{Ball}(x_0, \frac{D(x_0)}{2})$ where x_0 is an element of S. $C_1 \subseteq 2^{\alpha}$, where α is the carrier of S_{top}. For every subset P of S_{top} such that $P \in C_1$ holds P is open. The carrier of $S_{\mathrm{top}} \subseteq \bigcup C_1$. Consider G being a family of subsets of S_{top} such that $G \subseteq C_1$ and G is cover of $\Omega_{S_{\mathrm{top}}}$ and finite. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point x_0 of S such that $\$_2 = x_0$ and $\$_1 = \mathrm{Ball}(x_0, \frac{D(x_0)}{2})$. For every object Z such that $Z \in G$ there exists an object x_0 such that $x_0 \in$ the carrier of S and $\mathcal{Q}[Z, x_0]$.

Consider H being a function from G into the carrier of S such that for every object Z such that $Z \in G$ holds $\mathcal{Q}[Z, H(Z)]$. For every object Z such that $Z \in G$ holds $Z = \mathrm{Ball}(H_{/Z}, \frac{D(H(Z))}{2})$. Reconsider $D_0 = D^{\circ}(\mathrm{rng}\, H)$ as a finite subset of $\mathbb{R}$. $G \neq \emptyset$. Consider x_3 being an object such that $x_3 \in G$. Consider x_3 being an object such that $x_3 \in \mathrm{rng}\, H$. Set $d_0 = \inf D_0$. Consider x_3 being an object such that $x_3 \in \mathrm{dom}\, D$ and $x_3 \in \mathrm{rng}\, H$ and $d_0 = D(x_3)$. For every function f from S into T such that $f \in F$ for every points x_1, x_2 of S such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. □

2. Ascoli-Arzelà Theorem

From now on S, Z denote real normed spaces, T denotes a real Banach space, and F denotes a subset of (the carrier of $T)^{(\text{the carrier of } S)}$.

Now we state the proposition:

(6) Let us consider a real normed space Z. Then Z is complete if and only if MetricSpaceNorm Z is complete.

PROOF: For every sequence s of Z such that s is Cauchy sequence by norm holds s is convergent by [10, (8)], [4, (5)]. □

Let us consider a real normed space Z and a non empty subset H of MetricSpaceNorm Z. Now we state the propositions:

(7) If Z is complete, then MetricSpaceNorm $Z\restriction\overline{H}$ is complete.
PROOF: Reconsider $F = H$ as a non empty subset of Z. $\overline{F} = \overline{H}$. Set $N =$ MetricSpaceNorm $Z\restriction\overline{H}$. For every sequence S_2 of N such that S_2 is Cauchy holds S_2 is convergent. □

(8) MetricSpaceNorm $Z\restriction H$ is totally bounded if and only if MetricSpaceNorm $Z\restriction\overline{H}$ is totally bounded.
PROOF: Reconsider $F = H$ as a non empty subset of Z. Consider D being a subset of (MetricSpaceNorm $Z)_{\rm top}$ such that $D = H$ and $\overline{H} = \overline{D}$. $\overline{F} = \overline{H}$. MetricSpaceNorm $Z\restriction H$ is totally bounded. □

(9) Let us consider a real normed space Z, a non empty subset F of Z, and a non empty subset H of MetricSpaceNorm Z. Suppose Z is complete and $H = F$ and MetricSpaceNorm $Z\restriction H$ is totally bounded. Then

(i) $\overline{H}$ is sequentially compact, and

(ii) MetricSpaceNorm $Z\restriction\overline{H}$ is compact, and

(iii) $\overline{F}$ is compact.

The theorem is a consequence of (1), (7), and (8).

(10) Let us consider a real normed space Z, a non empty subset F of Z, a non empty subset H of MetricSpaceNorm Z, and a subset T of TopSpaceNorm Z. Suppose Z is complete and $H = F$ and $H = T$. Then

(i) MetricSpaceNorm $Z\restriction H$ is totally bounded iff $\overline{H}$ is sequentially compact, and

(ii) MetricSpaceNorm $Z\restriction H$ is totally bounded iff MetricSpaceNorm $Z\restriction\overline{H}$ is compact, and

(iii) MetricSpaceNorm $Z\restriction H$ is totally bounded iff $\overline{F}$ is compact, and

(iv) MetricSpaceNorm $Z\restriction H$ is totally bounded iff $\overline{T}$ is compact.

The theorem is a consequence of (1), (7), and (8).

(11) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Let us consider a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T).

Then $\overline{H}$ is sequentially compact if and only if MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and $T)\restriction H$ is totally bounded. The theorem is a consequence of (7) and (8).

(12) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Let us consider a non

empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Suppose $H = F$. Then $\overline{F}$ is compact if and only if MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. The theorem is a consequence of (1) and (11).

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of T)$^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Now we state the propositions:

(13) Suppose $S = M_{\rm top}$ and T is complete. Then suppose $G = H$ and MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. Then G is equibounded and equicontinuous.

PROOF: Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. Set $M_1 =$ MetricSpaceNorm $Z\restriction H$. Consider L being a family of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \mathrm{Ball}(w, 1)$.

Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point w of M_1 such that $\$_2 = w$ and $\$_1 = \mathrm{Ball}(w, 1)$. For every object D such that $D \in L$ there exists an object w such that $w \in$ the carrier of M_1 and $\mathcal{Q}[D, w]$. Consider U being a function from L into the carrier of M_1 such that for every object D such that $D \in L$ holds $\mathcal{Q}[D, U(D)]$. For every object D such that $D \in L$ holds $D = \mathrm{Ball}(U_{/D}, 1)$. Set $N_1 =$ the norm of Z. Reconsider $N_2 = N_1{}^\circ(\mathrm{rng}\, U)$ as a finite subset of $\mathbb{R}$. Consider x_3 being an object such that $x_3 \in L$. Consider x_3 being an object such that $x_3 \in \mathrm{rng}\, U$. Set $d_0 = \sup N_2$. Set $K = d_0 + 1$.

For every function f from the carrier of M into the carrier of T such that $f \in G$ for every element x of M, $\|f(x)\| \leqslant K$. For every real number e such that $0 < e$ there exists a real number d such that $0 < d$ and for every function f from the carrier of M into the carrier of T such that $f \in G$ for every points x_1, x_2 of M such that $\rho(x_1, x_2) < d$ holds $\|f(x_1) - f(x_2)\| < e$. $\square$

(14) Suppose $S = M_{\rm top}$ and T is complete. Then suppose $G = H$ and MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T)$\restriction H$ is totally bounded. Then

(i) for every point x of S and for every non empty subset H_2 of MetricSpa–ceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T\restriction H_2$ is totally bounded, and

(ii) G is equicontinuous.

PROOF: For every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction H_2$ is totally bounded. □

(15) Let us consider a normed linear topological space T, and a real normed space R. Suppose $R =$ the normed structure of T and the topology of $T =$ the topology of TopSpaceNorm R. Then

(i) the distance by norm of $R =$ the distance by norm of T, and

(ii) MetricSpaceNorm $R =$ MetricSpaceNorm T, and

(iii) TopSpaceNorm $T =$ TopSpaceNorm R.

PROOF: For every points x, y of R, (the distance by norm of $T)(x, y) = \|x - y\|$ by [11, (19)]. □

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$, and a non empty subset H of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Now we state the propositions:

(16) Suppose $S = M_{\text{top}}$ and T is complete and $G = H$. Then MetricSpaceNo– rm(the $\mathbb{R}$-norm space of continuous functions of S and $T) \restriction H$ is totally bounded if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction \overline{H_2}$ is compact.

PROOF: Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. Set $M_1 =$ MetricSpaceNorm $Z \restriction H$. For every real number e such that $e > 0$ there exists a family L of subsets of M_1 such that L is finite and the carrier of $M_1 = \bigcup L$ and for every subset C of M_1 such that $C \in L$ there exists an element w of M_1 such that $C = \text{Ball}(w, e)$. □

(17) Suppose $S = M_{\text{top}}$ and T is complete and $G = H$. Then $\overline{H}$ is sequentially compact if and only if G is equicontinuous and for every point x of S and for every non empty subset H_2 of MetricSpaceNorm T such that $H_2 = \{f(x)$, where f is a function from S into $T : f \in H\}$ holds MetricSpaceNorm $T \restriction \overline{H_2}$ is compact. The theorem is a consequence of (11) and (16).

Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a subset G of (the carrier of $T)^{(\text{the carrier of } M)}$. Now we state the propositions:

(18) Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if G is equicontinuous and for every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. The theorem is a consequence of (12) and (16).

(19) Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$. Then $\overline{F}$ is compact if and only if for every point x of M, G is equicontinuous at x and for every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. The theorem is a consequence of (18) and (5).

(20) Let us consider a normed linear topological space T. Then T is compact if and only if TopSpaceNorm T is compact. The theorem is a consequence of (15).

(21) Let us consider a normed linear topological space T, and a set X. Then X is a compact subset of T if and only if X is a compact subset of TopSpaceNorm T. The theorem is a consequence of (15).

(22) Let us consider a normed linear topological space T. If T is compact, then T is complete. The theorem is a consequence of (20) and (6).

Let us observe that every normed linear topological space which is compact is also complete.

Now we state the proposition:

(23) Let us consider a non empty metric space M, a non empty, compact topological space S, a normed linear topological space T, a compact subset U of T, a non empty subset F of the $\mathbb{R}$-norm space of continuous functions of S and T, and a subset G of (the carrier of $T)^{\alpha}$. Suppose $S = M_{\mathrm{top}}$ and T is complete and $G = F$ and for every function f such that $f \in F$ holds $\operatorname{rng} f \subseteq U$. Then

(i) if $\overline{F}$ is compact, then G is equibounded and equicontinuous, and

(ii) if G is equicontinuous, then $\overline{F}$ is compact,

where α is the carrier of M.

Proof: Reconsider $H = F$ as a non empty subset of MetricSpaceNorm(the $\mathbb{R}$-norm space of continuous functions of S and T). Set $Z =$ the $\mathbb{R}$-norm space of continuous functions of S and T. MetricSpaceNorm $Z \restriction H$ is totally bounded iff $\overline{F}$ is compact. For every point x of S and for every non empty subset F_1 of MetricSpaceNorm T such that $F_1 = \{f(x)$, where f is a function from S into $T : f \in F\}$ holds MetricSpaceNorm $T \restriction \overline{F_1}$ is compact. $\square$

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