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Real Vector Space and Related Notions[1]

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Summary. In this paper, we discuss the properties that hold in finite dimensional vector spaces and related spaces. In the Mizar language [1], [2], variables are strictly typed, and their type conversion requires a complicated process. Our purpose is to formalize that some properties of finite dimensional vector spaces are preserved in type transformations, and to contain the complexity of type transformations into this paper. Specifically, we show that properties such as algebraic structure, subsets, finite sequences and their sums, linear combination, linear independence, and affine independence are preserved in type conversions among `TOP-REAL(n)`, `REAL-NS(n)`, and `n-VectSp_over F_Real`. We referred to [4], [9], and [8] in the formalization.

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1. Common Properties Between Norm and Topology in Finite Dimensional Linear Spaces

From now on X denotes a set, n, m, k denote natural numbers, K denotes a field, f denotes an n-element, real-valued finite sequence, and M denotes a matrix over $\mathbb{R}_{\rm F}$ of dimension $n \times m$.

Now we state the propositions:

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(1) The RLS structure of $\mathcal{E}^n_{\mathrm{T}}$ = the RLS structure of $\langle \mathcal{E}^n, \|\cdot\|\rangle$.
PROOF: For every elements x, y of $\mathcal{R}^n$, $+_{\mathcal{E}^n}(x,y) = +_{\mathbb{R}^{\mathrm{Seg}\,n}}(x,y)$. For every element x of $\mathbb{R}$ and for every element y of $\mathcal{R}^n$, $\cdot_{\mathcal{E}^n}(x,y) = \cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\,n}}(x,y)$ by [3, (3)]. □

(2) $\mathcal{E}^n$ = MetricSpaceNorm$\langle \mathcal{E}^n, \|\cdot\|\rangle$.
PROOF: Set $X = \langle \mathcal{E}^n, \|\cdot\|\rangle$. For every elements x, y of $\mathcal{R}^n$, (the distance of $\mathcal{E}^n)(x,y)$ = (the distance by norm of $X)(x,y)$. □

(3) The topological structure of $\mathcal{E}^n_{\mathrm{T}}$ = TopSpaceNorm$\langle \mathcal{E}^n, \|\cdot\|\rangle$. The theorem is a consequence of (2).

(4) The carrier of $\mathcal{E}^n_{\mathrm{T}}$ = the carrier of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. The theorem is a consequence of (1).

(5) The carrier of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ = the carrier of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. The theorem is a consequence of (4).

(6) $0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\langle \mathcal{E}^n, \|\cdot\|\rangle}$. The theorem is a consequence of (1).

(7) Let us consider elements p, q of $\mathcal{E}^n_{\mathrm{T}}$, and elements f, g of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. If $p = f$ and $q = g$, then $p + q = f + g$. The theorem is a consequence of (1).

(8) Let us consider a real number r, an element q of $\mathcal{E}^n_{\mathrm{T}}$, and an element g of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. If $q = g$, then $r \cdot q = r \cdot g$. The theorem is a consequence of (1).

(9) Let us consider an element q of $\mathcal{E}^n_{\mathrm{T}}$, and an element g of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. If $q = g$, then $-q = -g$. The theorem is a consequence of (8).

(10) Let us consider elements p, q of $\mathcal{E}^n_{\mathrm{T}}$, and elements f, g of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. If $p = f$ and $q = g$, then $p - q = f - g$. The theorem is a consequence of (9) and (7).

Let us consider a set X and a natural number n.

(11) X is a linear combination of $\langle \mathcal{E}^n, \|\cdot\|\rangle$ if and only if X is a linear combination of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (4).

(12) X is a linear combination of $\langle \mathcal{E}^n, \|\cdot\|\rangle$ if and only if X is a linear combination of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. The theorem is a consequence of (11).

(13) Let us consider a linear combination L_5 of $\mathcal{E}^n_{\mathrm{T}}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. Suppose $L_2 = L_5$. Then the support of L_2 = the support of L_5.

(14) Let us consider a linear combination L_5 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\|\rangle$. Suppose $L_2 = L_5$. Then the support of L_2 = the support of L_5. The theorem is a consequence of (11).

Let us consider a set F. Now we state the propositions:

(15) F is a subset of $\mathcal{E}^n_{\rm T}$ if and only if F is a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(16) F is a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ if and only if F is a subset of the n-dimension vector space over $\mathbb{R}_{\rm F}$.

(17) F is a finite sequence of elements of $\mathcal{E}^n_{\rm T}$ if and only if F is a finite sequence of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(18) F is a function from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}$ if and only if F is a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$. The theorem is a consequence of (4).

(19) Let us consider a finite sequence F_2 of elements of $\mathcal{E}^n_{\rm T}$, a function f_1 from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}$, a finite sequence F_4 of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a function f_3 from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$. If $f_1 = f_3$ and $F_2 = F_4$, then $f_1 \cdot F_2 = f_3 \cdot F_4$. The theorem is a consequence of (4) and (8).

(20) Let us consider a finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a function f_1 from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$, a finite sequence F_4 of elements of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and a function f_3 from the n-dimension vector space over $\mathbb{R}_{\rm F}$ into $\mathbb{R}_{\rm F}$. If $f_1 = f_3$ and $F = F_4$, then $f_1 \cdot F = f_3 \cdot F_4$. The theorem is a consequence of (18), (4), and (19).

(21) Let us consider a finite sequence F_3 of elements of $\mathcal{E}^n_{\rm T}$, and a finite sequence F_2 of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $F_3 = F_2$, then $\sum F_3 = \sum F_2$.
Proof: Set $T = \mathcal{E}^n_{\rm T}$. Set $V = \langle \mathcal{E}^n, \|\cdot\| \rangle$. Consider f being a sequence of the carrier of T such that $\sum F = f(\operatorname{len} F)$ and $f(0) = 0_T$ and for every natural number j and for every element v of T such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

Consider f_3 being a sequence of the carrier of V such that $\sum F_4 = f_3(\operatorname{len} F_4)$ and $f_3(0) = 0_V$ and for every natural number j and for every element v of V such that $j < \operatorname{len} F_4$ and $v = F_4(j+1)$ holds $f_3(j+1) = f_3(j) + v$. Define $\mathcal{S}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $f(\$_1) = f_3(\$_1)$. For every natural number i such that $\mathcal{S}[i]$ holds $\mathcal{S}[i+1]$. $\mathcal{S}[0]$. For every natural number n, $\mathcal{S}[n]$. $\square$

(22) Let us consider a finite sequence F of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a finite sequence F_4 of elements of the n-dimension vector space over $\mathbb{R}_{\rm F}$. If $F_4 = F$, then $\sum F = \sum F_4$. The theorem is a consequence of (4) and (21).

(23) Let us consider a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a linear combination L_4 of $\mathcal{E}^n_{\rm T}$. If $L_2 = L_4$, then $\sum L_2 = \sum L_4$. The theorem is a consequence of (4), (19), and (21).

(24) Let us consider a linear combination L_5 of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and a linear combination L_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $L_2 = L_5$, then $\sum L_2 = \sum L_5$. The theorem is a consequence of (11) and (23).

(25) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\rm T}$. Suppose $A_3 = A_4$. Let us consider an object X. Then X is a linear combination

of A_3 if and only if X is a linear combination of A_4. The theorem is a consequence of (11).

(26) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_4)}$. The theorem is a consequence of (11) and (23).

(27) Let us consider a subset A_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. If $A_2 = A_3$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (4) and (26).

(28) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_3 = A_4$. Then A_3 is linearly independent if and only if A_4 is linearly independent. The theorem is a consequence of (11), (6), and (23).

(29) Let us consider a subset A_2 of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, and a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $A_2 = A_3$. Then A_2 is linearly independent if and only if A_3 is linearly independent. The theorem is a consequence of (4) and (28).

(30) Let us consider an object X. Then X is a subspace of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ if and only if X is a subspace of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (1), (4), and (6).

(31) Let us consider a set X, a subspace U of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a subspace W of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $\Omega_U = \Omega_W$. Then X is a linear combination of U if and only if X is a linear combination of W. The theorem is a consequence of (30).

(32) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\mathrm{rng}\, F$ is linearly independent. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that

   (i) M is invertible, and

   (ii) $M \restriction \mathrm{len}\, F = F$.

   The theorem is a consequence of (4) and (28).

(33) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n, and a square matrix N over $\mathbb{R}$ of dimension n. Suppose $N = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})M$. Then M is invertible if and only if N is invertible.

(34) Let us consider a square matrix M over $\mathbb{R}$ of dimension n. Then M is invertible if and only if $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M$ is invertible.

(35) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\mathrm{rng}\, F$ is linearly independent. Then there exists a square matrix M over $\mathbb{R}$ of dimension n such that

   (i) M is invertible, and

(ii) $M \restriction \operatorname{len} F = F$.

The theorem is a consequence of (32) and (33).

(36) Let us consider a one-to-one finite sequence F of elements of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\operatorname{rng} F$ is linearly independent. Let us consider an ordered basis B of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B = \mathrm{MX2FinS}\, I_{\mathbb{R}_{\mathrm{F}}}^{n \times n}$. Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n. Suppose M is invertible and $M \restriction \operatorname{len} F = F$. Then $(\mathrm{Mx2Tran}(M))^{\circ}(\Omega_{\mathrm{Lin}(\operatorname{rng}(B \restriction \operatorname{len} F))}) = \Omega_{\mathrm{Lin}(\operatorname{rng} F)}$. The theorem is a consequence of (4), (28), and (26).

(37) Let us consider linearly independent subsets A, B of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\overline{\overline{A}} = \overline{\overline{B}}$. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that

(i) M is invertible, and

(ii) $(\mathrm{Mx2Tran}(M))^{\circ}(\Omega_{\mathrm{Lin}(A)}) = \Omega_{\mathrm{Lin}(B)}$.

The theorem is a consequence of (4), (28), and (26).

(38) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and a linearly independent subset A of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{\circ} A$ is linearly independent. The theorem is a consequence of (4) and (28).

(39) Let us consider an element p of $\mathcal{E}^n_{\mathrm{T}}$, an element f of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, a subset H of $\mathcal{E}^n_{\mathrm{T}}$, and a subset I of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. If $p = f$ and $H = I$, then $p + H = f + I$. The theorem is a consequence of (4) and (7).

(40) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then A_3 is affine iff A_4 is affine. The theorem is a consequence of (4), (8), and (7).

(41) Let us consider a set X. Then X is an affinely independent subset of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ if and only if X is an affinely independent subset of $\mathcal{E}^n_{\mathrm{T}}$. The theorem is a consequence of (4), (6), (9), (39), and (28).

(42) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{\circ} A$ is affinely independent. The theorem is a consequence of (41).

(43) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a subset A_4 of $\mathcal{E}^n_{\mathrm{T}}$. If $A_3 = A_4$, then $\mathrm{Affin}\, A_3 = \mathrm{Affin}\, A_4$. The theorem is a consequence of (4) and (40).

(44) Let us consider a linear combination L of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and a linear combination S of $\mathcal{E}^n_{\mathrm{T}}$. If $L = S$, then $\operatorname{sum} L = \operatorname{sum} S$. The theorem is a consequence of (4).

(45) Let us consider a subset A_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, a subset A_4 of $\mathcal{E}^n_{\rm T}$, an element v of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and an element w of $\mathcal{E}^n_{\rm T}$. Suppose $A_3 = A_4$ and $v = w$ and $v \in \mathrm{Affin}\, A_3$ and A_3 is affinely independent. Then $v \rightarrow A_3 = w \rightarrow A_4$. The theorem is a consequence of (41), (25), (23), (44), and (43).

(46) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\rm F}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Let us consider an element v of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $v \in \mathrm{Affin}\, A$. Then

(i) $(\mathrm{Mx2Tran}(M))(v) \in \mathrm{Affin}((\mathrm{Mx2Tran}(M))^\circ A)$, and

(ii) for every n-element, real-valued finite sequence f, $(v \rightarrow A)(f) = ((\mathrm{Mx2Tran}(M))(v) \rightarrow (\mathrm{Mx2Tran}(M))^\circ A)((\mathrm{Mx2Tran}(M))(f))$.

The theorem is a consequence of (41), (4), (43), and (45).

(47) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\rm F}$ of dimension $n \times m$, and a linearly independent subset A of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{-1}(A)$ is linearly independent. The theorem is a consequence of (4) and (28).

(48) Let us consider natural numbers n, m, a matrix M over $\mathbb{R}_{\rm F}$ of dimension $n \times m$, and an affinely independent subset A of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $\mathrm{rk}(M) = n$. Then $(\mathrm{Mx2Tran}(M))^{-1}(A)$ is affinely independent. The theorem is a consequence of (41).

(49) Let us consider a real linear space V. Then every strict subspace of V is a strict subspace of Ω_V.

(50) Let us consider a set X. Then X is a basis of the n-dimension vector space over $\mathbb{R}_{\rm F}$ if and only if X is a basis of $\mathcal{E}^n_{\rm T}$.

Let us consider a non empty natural number n.

(51) $+_{\mathbb{R}^{\mathrm{Seg}\, n}} = \pi^n(\text{the addition of } \mathbb{R}_{\rm F})$.

PROOF: Set $O_1 = +_{\mathbb{R}^{\mathrm{Seg}\, n}}$. Set $O_2 = \pi^n(\text{the addition of } \mathbb{R}_{\rm F})$. For every elements x, y of $\mathcal{R}^n$, $O_1(x, y) = O_2(x, y)$. □

(52) $\cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\, n}} = \cdot^{n}_{\mathbb{R}_{\rm F}}$.

PROOF: Set $O_1 = \cdot^{\mathbb{R}}_{\mathbb{R}^{\mathrm{Seg}\, n}}$. Set $O_2 = \cdot^{n}_{\mathbb{R}_{\rm F}}$. For every element x of $\mathbb{R}$ and for every element y of $\mathcal{R}^n$, $O_1(x, y) = O_2(x, y)$. □

(53) (i) $\mathcal{E}^n_{\rm T}$ is finite dimensional, and

(ii) $\dim(\mathcal{E}^n_{\rm T}) = n$.

The theorem is a consequence of (50).

(54) Let us consider a non empty natural number n. Then

(i) the carrier of $\mathcal{E}^n_{\rm T}$ = the carrier of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and

(ii) $0_{\mathcal{E}^n_{\rm T}} = 0_\alpha$, and

(iii) the addition of $\mathcal{E}^n_{\rm T}$ = the addition of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and

(iv) the external multiplication of $\mathcal{E}^n_{\rm T}$ = the left multiplication of the n-dimension vector space over $\mathbb{R}_{\rm F}$,

where α is the n-dimension vector space over $\mathbb{R}_{\rm F}$. The theorem is a consequence of (51) and (52).

(55) Let us consider a non empty natural number n, elements x_2, y_2 of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and elements x_1, y_1 of $\mathcal{E}^n_{\rm T}$. If $x_2 = x_1$ and $y_2 = y_1$, then $x_2 + y_2 = x_1 + y_1$.

(56) Let us consider a non empty natural number n, an element a_1 of $\mathbb{R}_{\rm F}$, a real number a_2, an element x_2 of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and an element x_1 of $\mathcal{E}^n_{\rm T}$. If $a_1 = a_2$ and $x_2 = x_1$, then $a_1 \cdot x_2 = a_2 \cdot x_1$.

(57) Let us consider a non empty natural number n, an element x_2 of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and an element x_1 of $\mathcal{E}^n_{\rm T}$. If $x_2 = x_1$, then $-x_2 = -x_1$. The theorem is a consequence of (54).

(58) Let us consider a non empty natural number n, elements x_2, y_2 of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and elements x_1, y_1 of $\mathcal{E}^n_{\rm T}$. If $x_2 = x_1$ and $y_2 = y_1$, then $x_2 - y_2 = x_1 - y_1$. The theorem is a consequence of (57) and (54).

(59) Let us consider a non empty natural number n, a subset A_4 of $\mathcal{E}^n_{\rm T}$, and a subset A_3 of the n-dimension vector space over $\mathbb{R}_{\rm F}$. Suppose $A_4 = A_3$. Then

(i) the carrier of $\mathrm{Lin}(A_4)$ = the carrier of $\mathrm{Lin}(A_3)$, and

(ii) $0_{\mathrm{Lin}(A_4)} = 0_{\mathrm{Lin}(A_3)}$, and

(iii) the addition of $\mathrm{Lin}(A_4)$ = the addition of $\mathrm{Lin}(A_3)$, and

(iv) the external multiplication of $\mathrm{Lin}(A_4)$ = the left multiplication of $\mathrm{Lin}(A_3)$.

The theorem is a consequence of (54).

(60) Let us consider a subset A_4 of $\mathcal{E}^n_{\rm T}$, and a subset A_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. If $A_4 = A_3$, then $\mathrm{Lin}(A_4) = \mathrm{Lin}(A_3)$. The theorem is a consequence of (26) and (1).

(61) Let us consider a set X. Then X is a basis of $\mathcal{E}^n_{\rm T}$ if and only if X is a basis of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. The theorem is a consequence of (4), (28), (49), and (26).

(62) (i) $\langle \mathcal{E}^n, \| \cdot \| \rangle$ is finite dimensional, and

(ii) $\dim(\langle \mathcal{E}^n, \| \cdot \| \rangle) = n$.

The theorem is a consequence of (53), (4), and (61).

2. Finite Dimensional Vector Spaces over Real Field

Note that there exists a real normed space which is finite dimensional.

Now we state the propositions:

(63) Let us consider a field K, a finite dimensional vector space V over K, and an ordered basis b of V. Then there exists a linear transformation T from V to the dim(V)-dimension vector space over K such that

(i) T is bijective, and

(ii) for every element x of V, $T(x) = x \rightarrow b$.

Proof: Set W = the dim(V)-dimension vector space over K. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an element x of V such that $\$_1 = x$ and $\$_2 = x \rightarrow b$.

For every element x of the carrier of V, there exists an element y of the carrier of W such that $\mathcal{P}[x, y]$. Consider f being a function from the carrier of V into the carrier of W such that for every element x of the carrier of V, $\mathcal{P}[x, f(x)]$. For every element x of V, $f(x) = x \rightarrow b$. For every elements x, y of V, $f(x+y) = f(x) + f(y)$. For every scalar a of K and for every vector x of V, $f(a \cdot x) = a \cdot f(x)$. For every objects x, y such that $x, y \in \operatorname{dom} f$ and $f(x) = f(y)$ holds $x = y$.

For every object y such that $y \in$ the carrier of W there exists an object x such that $x \in$ the carrier of V and $y = f(x)$ by [6, (102)], [7, (21)], [5, (36)]. □

(64) Let us consider a field K, and a finite dimensional vector space V over K. Then there exists a linear transformation T from V to the dim(V)-dimension vector space over K such that T is bijective. The theorem is a consequence of (63).

(65) Let us consider a field K, and finite dimensional vector spaces V, W over K. Then $\dim(V) = \dim(W)$ if and only if there exists a linear transformation T from V to W such that T is bijective. The theorem is a consequence of (64).

(66) Let us consider a real linear space X. Then

(i) the carrier of X = the carrier of RLSp2RVSp(X), and

(ii) the zero of X = the zero of RLSp2RVSp(X), and

(iii) the addition of X = the addition of RLSp2RVSp(X), and

(iv) the external multiplication of X =
the left multiplication of RLSp2RVSp(X).

(67) Let us consider a strict real linear space X.
Then RVSp2RLSp RLSp2RVSp(X) = X.

(68) Let us consider a strict vector space X over $\mathbb{R}_{\mathrm{F}}$.
Then RLSp2RVSp(RVSp2RLSp X) $= X$.

Let us consider a real linear space V and a set F.

(69) F is a subset of V if and only if F is a subset of RLSp2RVSp(V).

(70) F is a finite sequence of elements of V if and only if F is a finite sequence of elements of RLSp2RVSp(V).

(71) F is a function from V into $\mathbb{R}$ if and only if F is a function from RLSp2RVSp(V) into $\mathbb{R}$.

(72) Let us consider a real linear space T, and a set X. Then X is a linear combination of RLSp2RVSp(T) if and only if X is a linear combination of T.

(73) Let us consider a real linear space T, a linear combination L_5 of RLSp2RVSp(T), and a linear combination L_2 of T. Suppose $L_2 = L_5$. Then the support of $L_2 =$ the support of L_5.
Proof: The support of $L_2 \subseteq$ the support of L_5. Consider u being an element of RLSp2RVSp(T) such that $x = u$ and $L_5(u) \neq 0_{\mathbb{R}_{\mathrm{F}}}$. □

(74) Let us consider a real linear space V, a finite sequence F_2 of elements of V, a function f_1 from V into $\mathbb{R}$, a finite sequence F_4 of elements of RLSp2RVSp(V), and a function f_3 from RLSp2RVSp(V) into $\mathbb{R}_{\mathrm{F}}$. If $f_1 = f_3$ and $F_2 = F_4$, then $f_1 \cdot F_2 = f_3 \cdot F_4$.

(75) Let us consider a real linear space T, a finite sequence F_3 of elements of T, and a finite sequence F_2 of elements of RLSp2RVSp(T). If $F_3 = F_2$, then $\sum F_3 = \sum F_2$.

(76) Let us consider a real linear space T, a linear combination L_5 of RLSp2RVSp(T), and a linear combination L_2 of T. If $L_2 = L_5$, then $\sum L_2 = \sum L_5$. The theorem is a consequence of (73) and (74).

Let us consider a real linear space T, a subset A_2 of RLSp2RVSp(T), and a subset A_3 of T. Now we state the propositions:

(77) If $A_2 = A_3$, then $\Omega_{\mathrm{Lin}(A_3)} = \Omega_{\mathrm{Lin}(A_2)}$. The theorem is a consequence of (72), (73), and (76).

(78) If $A_2 = A_3$, then A_2 is linearly independent iff A_3 is linearly independent. The theorem is a consequence of (72), (73), and (76).

(79) Let us consider a real linear space T, a set X, a subspace U of RLSp2RVSp(T), and a subspace W of T. Suppose $\Omega_U = \Omega_W$. Then X is a linear combination of U if and only if X is a linear combination of W.

(80) Let us consider a real linear space W, and a set X. Then X is a basis of RLSp2RVSp(W) if and only if X is a basis of W. The theorem is a consequence of (78) and (77).

Let us consider a real linear space W. Now we state the propositions:

(81) If W is finite dimensional, then RLSp2RVSp(W) is finite dimensional and dim(RLSp2RVSp(W)) = dim(W). The theorem is a consequence of (80).

(82) W is finite dimensional if and only if RLSp2RVSp(W) is finite dimensional. The theorem is a consequence of (80).

(83) Let us consider a non empty natural number n. Then RLSp2RVSp($\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\, n}$) = the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. The theorem is a consequence of (51) and (52).

(84) Let us consider real linear spaces V, W, and a set X. Then X is a linear operator from V into W if and only if X is a linear transformation from RLSp2RVSp(V) to RLSp2RVSp(W).

(85) Let us consider real linear spaces X, Y, and a linear operator L from X into Y. Suppose L is bijective. Then there exists a linear operator K from Y into X such that

(i) $K = L^{-1}$, and

(ii) K is one-to-one and onto.

PROOF: Reconsider $K = L^{-1}$ as a function from the carrier of Y into the carrier of X. K is additive. For every vector x of Y and for every real number r, $K(r \cdot x) = r \cdot K(x)$. □

(86) Let us consider real linear spaces X, Y, Z, a linear operator L from X into Y, and a linear operator K from Y into Z. Then $K \cdot L$ is a linear operator from X into Z.

PROOF: Reconsider $T = K \cdot L$ as a function from X into Z. For every elements x, y of X, $T(x + y) = T(x) + T(y)$. For every real number a and for every vector x of X, $T(a \cdot x) = a \cdot T(x)$. □

(87) Let us consider real linear spaces V, W, a subset A of V, and a linear operator T from V into W. Suppose T is bijective. Then A is a basis of V if and only if $T^{\circ}A$ is a basis of W. The theorem is a consequence of (84) and (80).

(88) Let us consider a finite dimensional real linear space V, and a real linear space W. Suppose there exists a linear operator T from V into W such that T is bijective. Then

(i) W is finite dimensional, and

(ii) dim(W) = dim(V).

The theorem is a consequence of (87).

(89) Let us consider a finite dimensional real linear space V. Suppose dim(V)

$\neq 0$. Then there exists a linear operator T from V into $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,\dim(V)}$ such that T is bijective. The theorem is a consequence of (81), (64), (83), and (84).

(90) Let us consider finite dimensional real linear spaces V, W. Suppose $\dim(V) \neq 0$. Then $\dim(V) = \dim(W)$ if and only if there exists a linear operator T from V into W such that T is bijective. The theorem is a consequence of (89), (85), (86), and (88).

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