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On Weakly Associative Lattices and Near Lattices

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Summary. The main aim of this article is to introduce formally two generalizations of lattices, namely weakly associative lattices and near lattices, which can be obtained from the former by certain weakening of the usual well-known axioms. We show selected propositions devoted to weakly associative lattices and near lattices from Chapter 6 of [15], dealing also with alternative versions of classical axiomatizations. Some of the results were proven in the Mizar [1], [2] system with the help of Prover9 [14] proof assistant.

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0. Introduction

Lattice theory is widely represented in the Mizar Mathematical Library, with Żukowski's first article [18], following Birkhoff [3] and Grätzer [11], [12]. In parallel, the theory of partially ordered sets was developed [4] treated generally as relational structures, even if informally the notions are quite similar [9], [7]. The review of the mechanization of lattice theory in Mizar, with the example of the solution of the Robbins problem, is contained in [6].

Our work can be seen as a step towards a Mizar support for [15] or [16], where original proof objects by OTTER/Prover9 were used. Some preliminary works in this direction were already done in [8] by present authors. We use the interface `ott2miz` [17] which allows for the automated translation of proofs;

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these automatically generated proofs are usually quite lengthy, even after native enhancements done by internal Mizar software for library revisions.

Weakly associative lattices were studied in [5]. In the present development, we deal with the parts of Chap. 6 "Lattice-like algebras" of [15], pp. 111–135, devoted to this class of lattices. In this sense, we continue the work started by Kulesza and Grabowski in [13], devoted to the formalization of quasi-lattices.

The class of weakly associative lattices (or WA-lattices, WAL) can be characterized from the standard set of axioms for lattices (with idempotence for the join and meet operations included), where the ordinary associative laws are replaced by the so-called part-preservation laws. The characteristic axiom is however W3 (or, dual W3' – compare Def. 1 and Def. 2). Section 2 contains also equivalent formulation of these axioms, using ordering relation on lattices. The earlier seems to be a bit more feasible taking into account the role of equality in the Mizar system [10] and the design of Prover9.

In Section 3 we show how described binary lattice operations can be associated with the corresponding ordering relation. Obviously, the associativity can only be shown under some conditions for elements (see theorems (15) and (16)). If we assume distributivity, the relation is transitive, as in usual lattices. Section 4 contains the proof that adding the distributivity condition to the set of usual WAL axioms, the associativity can be proven.

Then we deal with another generalization of lattices, i.e. near lattices (absorption law is weakened). Def. 6 and Def. 7 provide standard examples of these structures which are not necessarily lattices (see Def. 10 for the definition of the structure). The lattice operations are given by

$\sqcup$	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

$\sqcap$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

Associativity laws do not hold here, so this is not a lattice.

The rest of the article is devoted to alternative axiomatizations of WAL. WAL-3 – equivalent set of axioms describing WAL is expressed in the form of five separate attributes to make Mizar registrations mechanism working (see Def. 11–Def. 15). It is shown that these adjectives imply the standard attributes for lattices.

In Section 8 WAL-4 is defined (the short sigle axiom system for WAL). We conclude with the proof if WAL-4 holds, then lattice operations are idempotent.

Some of the proofs were produced by means of Prover9, so they are definitely lengthy. The enhancement of the lemmas, including their shortening, reorganization and clustering, can be interesting and useful future work.

1. Preliminaries

From now on L denotes a non empty lattice structure and v_{100}, v_{102}, v_2, v_1, v_0, v_3, v_{101} denote elements of L.

Let us consider v_0, v_1, and v_2. Now we state the propositions:

(1) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$.

(2) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$.

Let us consider v_1 and v_2. Now we state the propositions:

(3) Suppose for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1, v_2, and v_0, $v_0 \sqcap (v_1 \sqcup (v_0 \sqcup v_2)) = v_0$. Then $v_1 \sqcap (v_1 \sqcup v_2) = v_1$.

(4) Suppose for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$. Then $v_1 \sqcup (v_1 \sqcap v_2) = v_1$.

2. Definition of Attributes

Let L be a non empty lattice structure. We say that L is satisfying W3 if and only if

(Def. 1) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$.

We say that L is satisfying W3' if and only if

(Def. 2) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$.

Let L be a meet-absorbing, join-absorbing, meet-commutative, non empty lattice structure. Let us note that L is satisfying W3 if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every elements v_2, v_1, v_0 of L, $v_1 \sqsubseteq (v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)$.

Let us consider L. Observe that L is satisfying W3' if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1) \sqsubseteq v_1$.

Let us note that every non empty lattice structure which is meet-commutative, join-idempotent, join-commutative, and satisfying W3' is also quasi-meet-absorbing and every non empty lattice structure which is meet-commutative, meet-idempotent, join-commutative, and satisfying W3 is also join-absorbing and every non empty lattice structure which is trivial is also satisfying W3' and there exists a non empty lattice structure which is satisfying W3, satisfying W3', join-idempotent, meet-idempotent, join-commutative, and meet-commutative.

A weakly associative lattice is a join-idempotent, meet-idempotent, join-commutative, meet-commutative, satisfying W3, satisfying W3', non empty lattice structure.

A WA-lattice is a weakly associative lattice. Note that every join-associative, meet-absorbing lattice is satisfying W3'.

Let L be a non empty lattice structure. We say that L is satisfying WA if and only if

(Def. 5) for every elements x, y, z of L, $x \sqcap (y \sqcup (x \sqcup z)) = x$.

3. On the Ordering Relation Generated by Weakly Associated Lattices

Let us note that every non empty lattice structure which is quasi-meet-absorbing, meet-commutative, and join-commutative is also meet-absorbing and every WA-lattice is meet-absorbing.

From now on L denotes a WA-lattice and x, y, z, u denote elements of L.

Now we state the propositions:

(5) $x \sqcup y = y$ if and only if $x \sqsubseteq y$.

(6) $x \sqcap y = x$ if and only if $x \sqsubseteq y$.

(7) $x \sqsubseteq x$.

(8) If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$.

(9) $x \sqsubseteq x \sqcup y$.

(10) $x \sqcap y \sqsubseteq x$.

(11) If $x \sqsubseteq z$ and $y \sqsubseteq z$, then $x \sqcup y \sqsubseteq z$.

(12) There exists z such that

 (i) $x \sqsubseteq z$, and

 (ii) $y \sqsubseteq z$, and

 (iii) for every u such that $x \sqsubseteq u$ and $y \sqsubseteq u$ holds $z \sqsubseteq u$.

 The theorem is a consequence of (11) and (9).

(13) If $z \sqsubseteq x$ and $z \sqsubseteq y$, then $z \sqsubseteq x \sqcap y$.

(14) There exists z such that

(i) $z \sqsubseteq x$, and

(ii) $z \sqsubseteq y$, and

(iii) for every u such that $u \sqsubseteq x$ and $u \sqsubseteq y$ holds $u \sqsubseteq z$.

The theorem is a consequence of (13) and (10).

(15) If $x \sqsubseteq z$ and $y \sqsubseteq z$, then $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$.

(16) If $z \sqsubseteq x$ and $z \sqsubseteq y$, then $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$.

(17) If L is distributive and $x \sqsubseteq y \sqsubseteq z$, then $x \sqsubseteq z$.

4. Distributivity Implies Associativity

From now on L denotes a non empty lattice structure and v_0, v_1, v_2 denote elements of L.

Now we state the proposition:

(18) Suppose for every v_0, $v_0 \sqcap v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcap v_1 = v_1 \sqcap v_0$ and for every v_0, $v_0 \sqcup v_0 = v_0$ and for every v_1 and v_0, $v_0 \sqcup v_1 = v_1 \sqcup v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_1)) \sqcap v_1 = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_1)) \sqcup v_1 = v_1$ and for every v_1 and v_0, $v_0 \sqcap (v_0 \sqcup v_1) = v_0$ and for every v_0, v_2, and v_1, $v_0 \sqcup (v_1 \sqcap v_2) = (v_0 \sqcup v_1) \sqcap (v_0 \sqcup v_2)$. $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$.

Observe that every WA-lattice which is distributive' is also join-associative.

5. Near Lattices

Let x, y be elements of $\{0, 1, 2\}$. The functors: $x \sqcap_{\mathrm{NL}} y$ and $x \sqcup_{\mathrm{NL}} y$ yielding elements of $\{0, 1, 2\}$ are defined by terms

(Def. 6) $\begin{cases} 2, & \textbf{if } x = 0 \text{ and } y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ \min(x, y), & \textbf{otherwise}, \end{cases}$

(Def. 7) $\begin{cases} 0, & \textbf{if } x = 0 \text{ and } y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ \max(x, y), & \textbf{otherwise}, \end{cases}$

respectively. The functors: $\sqcup_{\mathrm{NL}}$ and $\sqcap_{\mathrm{NL}}$ yielding binary operations on $\{0, 1, 2\}$ are defined by conditions

(Def. 8) for every elements x, y of $\{0, 1, 2\}$, $\sqcup_{\mathrm{NL}}(x, y) = x \sqcup_{\mathrm{NL}} y$,

(Def. 9) for every elements x, y of $\{0, 1, 2\}$, $\sqcap_{\mathrm{NL}}(x, y) = x \sqcap_{\mathrm{NL}} y$,

respectively.

6. Examples of Near Lattices

The functor ExNearLattice yielding a non empty lattice structure is defined by the term

(Def. 10) $\langle\{0, 1, 2\}, \sqcup_{\mathrm{N}L}, \sqcap_{\mathrm{N}L}\rangle$.

One can check that ExNearLattice is non join-associative and non meet-associative and every non empty lattice structure which is trivial is also meet-idempotent, join-commutative, quasi-meet-absorbing, and join-absorbing.

A near lattice is a join-idempotent, meet-idempotent, join-commutative, meet-commutative, quasi-meet-absorbing, join-absorbing, non empty lattice structure.

One can check that ExNearLattice is join-commutative, meet-commutative, join-idempotent, meet-idempotent, join-absorbing, and meet-absorbing and every join-commutative, meet-commutative, non empty lattice structure which is meet-absorbing is also quasi-meet-absorbing and every join-commutative, meet-commutative, non empty lattice structure which is quasi-meet-absorbing is also meet-absorbing.

Now we state the proposition:

(19) (i) ExNearLattice is a near lattice, and

(ii) ExNearLattice is not a lattice.

7. Alternative Axioms for WAL

From now on L denotes a non empty lattice structure and v_{101}, v_{100}, v_2, v_1, v_0, v_{102}, v_{103}, v_3 denote elements of L.

Now we state the proposition:

(20) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. $v_0 \sqcup v_0 = v_0$.

Let us consider v_0 and v_1. Now we state the propositions:

(21) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. Then $v_0 \sqcap v_1 = v_1 \sqcap v_0$. The theorem is a consequence of (24).

(22) Suppose for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$ and for every v_0 and v_1, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$ and for every v_1 and v_0, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$ and for every v_2, v_1, and v_0, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$ and for every v_2, v_1, and v_0, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$. Then $v_0 \sqcup v_1 = v_1 \sqcup v_0$. The theorem is a consequence of (24) and (21).

Let L be a non empty lattice structure. We say that L is satisfying WAL-3_1 if and only if

(Def. 11) for every elements v_1, v_0 of L, $(v_0 \sqcap v_1) \sqcup (v_0 \sqcap (v_0 \sqcup v_1)) = v_0$.

We say that L is satisfying WAL-3_2 if and only if

(Def. 12) for every elements v_0, v_1 of L, $(v_0 \sqcap v_0) \sqcup (v_1 \sqcap (v_0 \sqcup v_0)) = v_0$.

We say that L is satisfying WAL-3_3 if and only if

(Def. 13) for every elements v_1, v_0 of L, $(v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1)) = v_1$.

We say that L is satisfying WAL-3_4 if and only if

(Def. 14) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcup v_1) \sqcap (v_2 \sqcup v_0)) \sqcap v_0 = v_0$.

We say that L is satisfying WAL-3_5 if and only if

(Def. 15) for every elements v_2, v_1, v_0 of L, $((v_0 \sqcap v_1) \sqcup (v_2 \sqcap v_0)) \sqcup v_0 = v_0$.

Let us note that every non empty lattice structure which is trivial is also satisfying WAL-3_1, satisfying WAL-3_2, satisfying WAL-3_3, satisfying WAL-3_4, and satisfying WAL-3_5 and every non empty lattice structure which is satisfying WAL-3_1, satisfying WAL-3_2, satisfying WAL-3_3, satisfying WAL-3_4, and satisfying WAL-3_5 is also join-idempotent, meet-idempotent, join-commutative, and meet-commutative.

8. Short Single Axiom for WAL

Let L be a non empty lattice structure. We say that L is satisfying WAL-4 if and only if

(Def. 16) for every elements v_2, v_0, v_5, v_4, v_3, v_1 of L, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$.

From now on L denotes a non empty lattice structure and v_{108}, v_{107}, v_{106}, v_{101}, v_{10}, v_9, v_8, v_7, v_6, v_{105}, v_{102}, v_{100}, v_{104}, v_{103}, v_5, v_4, v_3, v_2, v_1, v_0 denote elements of L.

Let us consider v_0. Now we state the propositions:

(23) Suppose for every v_2, v_0, v_5, v_4, v_3, and v_1, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap

$v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$. Then $v_0 \sqcap v_0 = v_0$.

(24) Suppose for every v_2, v_0, v_5, v_4, v_3, and v_1, $(((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcap v_2) \sqcup (((v_0 \sqcap (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)) \sqcup (((v_1 \sqcap (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)) \sqcup (v_5 \sqcap (v_1 \sqcup (((v_1 \sqcup v_3) \sqcap (v_4 \sqcup v_1)) \sqcap v_1)))) \sqcap (v_0 \sqcup (((v_1 \sqcap v_3) \sqcup (v_4 \sqcap v_1)) \sqcup v_1)))) \sqcap (((v_0 \sqcap v_1) \sqcup (v_1 \sqcap (v_0 \sqcup v_1))) \sqcup v_2)) = v_1$. Then $v_0 \sqcup v_0 = v_0$. The theorem is a consequence of (23).

One can check that every non empty lattice structure which is trivial is also satisfying WAL-4 and every non empty lattice structure which is satisfying WAL-4 is also join-idempotent and meet-idempotent.

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