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Pappus's Hexagon Theorem in Real Projective Plane[1]

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Summary. In this article we prove, using Mizar [2], [1], the Pappus's hexagon theorem in the real projective plane: "Given one set of collinear points A, B, C, and another set of collinear points a, b, c, then the intersection points X, Y, Z of line pairs Ab and aB, Ac and aC, Bc and bC are collinear"[2].

More precisely, we prove that the structure `ProjectiveSpace TOP-REAL3` [10] (where `TOP-REAL3` is a metric space defined in [5]) satisfies the Pappus's axiom defined in [11] by Wojciech Leończuk and Krzysztof Prażmowski. Eugeniusz Kusak and Wojciech Leończuk formalized the Hessenberg theorem early in the MML [9]. With this result, the real projective plane is Desarguesian.

For proving the Pappus's theorem, two different proofs are given. First, we use the techniques developed in the section "Projective Proofs of Pappus's Theorem" in the chapter "Pappos's Theorem: Nine proofs and three variations" [12]. Secondly, Pascal's theorem [4] is used.

In both cases, to prove some lemmas, we use `Prover9`[3], the successor of the `Otter` prover and `ott2miz` by Josef Urban[4] [13], [8], [7].

In `Coq`, the Pappus's theorem is proved as the application of Grassmann-Cayley algebra [6] and more recently in Tarski's geometry [3].

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[2]`https://en.wikipedia.org/wiki/Pappus's_hexagon_theorem`

[3]`https://www.cs.unm.edu/~mccune/prover9/`

[4]See its homepage `https://github.com/JUrban/ott2miz`

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1. Preliminaries

From now on a, b, c, d, e, f, g, h, i denote real numbers and M denotes a square matrix over $\mathbb{R}$ of dimension 3.

Now we state the propositions:

(1) Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Then $\mathrm{Det}\, M = a \cdot e \cdot i - c \cdot e \cdot g - a \cdot f \cdot h + b \cdot f \cdot g - b \cdot d \cdot i + c \cdot d \cdot h$.

(2) Let us consider elements P_1, P_4, P_5 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and elements p_1, p_2, p_3, p_4, p_5 of $\mathcal{E}^3_{\mathrm{T}}$. Suppose p_1 is not zero and $P_1 =$ the direction of p_1 and p_4 is not zero and $P_4 =$ the direction of p_4 and p_5 is not zero and $P_5 =$ the direction of p_5 and P_1, P_4 and P_5 are collinear. Then $\langle|p_1, p_2, p_4|\rangle \cdot \langle|p_1, p_3, p_5|\rangle = \langle|p_1, p_2, p_5|\rangle \cdot \langle|p_1, p_3, p_4|\rangle$.

(3) Let us consider non zero real numbers r_{416}, r_{415}, r_{413}, r_{418}, r_{419}, r_{412}, r_{712}, r_{746}, r_{716}, r_{742}, r_{715}, r_{743}, r_{713}, r_{745}, r_{749}, r_{718}, r_{719}, r_{748}. Suppose $(-r_{412})\cdot(-r_{713}) = (-r_{413})\cdot(-r_{712})$ and $(-r_{415})\cdot(-r_{719}) = (-r_{419})\cdot(-r_{715})$ and $(-r_{418}) \cdot (-r_{716}) = (-r_{416}) \cdot (-r_{718})$ and $(-r_{745}) \cdot r_{416} = (-r_{746}) \cdot r_{415}$ and $(-r_{748}) \cdot r_{413} = (-r_{743}) \cdot r_{418}$ and $(-r_{742}) \cdot r_{419} = (-r_{749}) \cdot r_{412}$ and $r_{712} \cdot r_{746} = r_{716} \cdot r_{742}$ and $r_{715} \cdot r_{743} = r_{713} \cdot r_{745}$. Then $r_{718} \cdot r_{749} = r_{719} \cdot r_{748}$.

2. Some Technical Lemmas Proved by `Prover9` and Translated with Help of `ott2miz`

From now on P_2 denotes a projective space defined in terms of collinearity and c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10} denote elements of P_2.

Now we state the propositions:

(4) Suppose $c_2 \neq c_1$ and $c_3 \neq c_1$ and $c_3 \neq c_2$ and $c_4 \neq c_2$ and $c_4 \neq c_3$ and $c_5 \neq c_1$ and $c_6 \neq c_1$ and $c_6 \neq c_5$ and $c_7 \neq c_5$ and $c_7 \neq c_6$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear. Then

(i) c_4, c_7 and c_2 are not collinear, and

(ii) c_4, c_{10} and c_3 are not collinear, and

(iii) c_4, c_7 and c_3 are not collinear, and

(iv) c_4, c_{10} and c_2 are not collinear, and

(v) c_4, c_7 and c_5 are not collinear, and

(vi) c_4, c_{10} and c_8 are not collinear, and

(vii) c_4, c_7 and c_8 are not collinear, and

(viii) c_4, c_{10} and c_5 are not collinear, and

(ix) c_4, c_7 and c_9 are not collinear, and

(x) c_4, c_{10} and c_6 are not collinear, and

(xi) c_4, c_7 and c_6 are not collinear, and

(xii) c_4, c_{10} and c_9 are not collinear, and

(xiii) c_7, c_{10} and c_5 are not collinear, and

(xiv) c_7, c_4 and c_6 are not collinear, and

(xv) c_7, c_{10} and c_9 are not collinear, and

(xvi) c_7, c_4 and c_3 are not collinear, and

(xvii) c_7, c_{10} and c_3 are not collinear, and

(xviii) c_7, c_4 and c_9 are not collinear, and

(xix) c_7, c_{10} and c_2 are not collinear, and

(xx) c_7, c_4 and c_8 are not collinear, and

(xxi) c_{10}, c_4 and c_2 are not collinear, and

(xxii) c_{10}, c_7 and c_6 are not collinear, and

(xxiii) c_{10}, c_4 and c_6 are not collinear, and

(xxiv) c_{10}, c_7 and c_2 are not collinear, and

(xxv) c_{10}, c_4 and c_5 are not collinear, and

(xxvi) c_{10}, c_7 and c_3 are not collinear, and

(xxvii) c_{10}, c_4 and c_3 are not collinear, and

(xxviii) c_{10}, c_7 and c_5 are not collinear.

(5) Suppose $c_2 \neq c_1$ and $c_3 \neq c_2$ and $c_5 \neq c_1$ and $c_7 \neq c_5$ and $c_7 \neq c_6$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear.
Then c_{10}, c_7 and c_8 are not collinear.

(6) Suppose c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear. Then

(i) c_4, c_2 and c_3 are collinear, and

(ii) c_4, c_5 and c_8 are collinear, and

(iii) c_4, c_9 and c_6 are collinear, and

(iv) c_7, c_5 and c_6 are collinear, and

(v) c_7, c_9 and c_3 are collinear, and

(vi) c_7, c_2 and c_8 are collinear, and

(vii) c_{10}, c_2 and c_6 are collinear, and

(viii) c_{10}, c_5 and c_3 are collinear.

(7) Suppose $c_3 \neq c_1$ and $c_3 \neq c_2$ and $c_6 \neq c_1$ and $c_6 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_5 and c_6 are collinear. Then

(i) c_2, c_3 and c_5 are not collinear, and

(ii) c_2, c_3 and c_6 are not collinear, and

(iii) c_2, c_5 and c_6 are not collinear, and

(iv) c_3, c_5 and c_6 are not collinear.

(8) Suppose $c_3 \neq c_1$ and $c_4 \neq c_1$ and $c_4 \neq c_3$ and $c_3 \neq c_2$ and $c_4 \neq c_2$ and $c_6 \neq c_1$ and $c_7 \neq c_1$ and $c_7 \neq c_6$ and $c_6 \neq c_5$ and $c_7 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear. Then

(i) c_1, c_3 and c_6 are not collinear, and

(ii) c_1, c_3 and c_4 are collinear, and

(iii) c_1, c_6 and c_7 are collinear, and

(iv) $c_3 \neq c_1$, and

(v) $c_2 \neq c_1$, and

(vi) $c_3 \neq c_2$, and

(vii) $c_4 \neq c_3$, and

(viii) $c_4 \neq c_2$, and

(ix) $c_6 \neq c_1$, and

(x) $c_5 \neq c_1$, and

(xi) $c_6 \neq c_5$, and

(xii) $c_7 \neq c_6$, and

(xiii) $c_7 \neq c_5$, and

(xiv) c_1, c_4 and c_7 are not collinear, and

(xv) c_1, c_4 and c_3 are collinear, and

(xvi) c_1, c_4 and c_2 are collinear, and

(xvii) c_1, c_7 and c_6 are collinear, and

(xviii) c_1, c_7 and c_5 are collinear.

(9) Suppose $c_4 \neq c_2$ and $c_4 \neq c_3$ and $c_8 \neq c_2$ and c_2, c_3 and c_6 are not collinear. Then

(i) c_2, c_3 and c_4 are not collinear, or

(ii) c_2, c_6 and c_8 are not collinear, or

(iii) c_3, c_4 and c_8 are not collinear.

(10) Suppose $c_4 \neq c_1$ and $c_6 \neq c_5$ and c_1, c_2 and c_5 are not collinear. Then

(i) c_1, c_2 and c_4 are not collinear, or

(ii) c_1, c_5 and c_6 are not collinear, or

(iii) c_4, c_6 and c_8 are not collinear, or

(iv) $c_8 \neq c_5$.

(11) Suppose $c_4 \neq c_2$ and $c_6 \neq c_1$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_4, c_6 and c_8 are collinear. Then $c_8 \neq c_2$.

(12) If c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear, then c_2, c_3 and c_4 are collinear.

(13) If c_1, c_2 and c_5 are not collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear, then c_5, c_6 and c_7 are collinear.

(14) If $c_3 \neq c_1$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_5 and c_7 are collinear, then $c_7 \neq c_3$.

(15) Suppose $c_4 \neq c_1$ and $c_4 \neq c_3$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_3 are collinear and c_1, c_2 and c_4 are collinear and c_4, c_5 and c_9 are collinear. Then $c_9 \neq c_3$.

(16) Suppose $c_4 \neq c_1$ and $c_4 \neq c_2$ and $c_6 \neq c_1$ and $c_7 \neq c_6$ and $c_7 \neq c_5$ and c_1, c_2 and c_5 are not collinear and c_1, c_2 and c_4 are collinear and c_1, c_5 and c_6 are collinear and c_1, c_5 and c_7 are collinear and c_2, c_7 and c_9 are collinear and c_4, c_5 and c_9 are collinear. Then c_9, c_2 and c_5 are not collinear.

3. The Real Projective Plane and Pappus's Theorem

From now on o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 denote elements of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Now we state the propositions:

(17) Pappus theorem as "Pappos's Theorem: Nine proofs and three variations" [12] version:
Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

(18) The projective space over $\mathcal{E}^3_{\mathrm{T}}$ is a Pappian, Desarguesian projective plane defined in terms of collinearity.

4. Proof: Special Case of Pascal's Theorem

In the sequel v_0, v_1, v_2, v_3, v_4, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, v_{100}, v_{101}, v_{102}, v_{103} denote elements of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.
Now we state the propositions:

(19) Suppose $c_1 \neq c_2$ and $c_1 \neq c_3$ and $c_2 \neq c_3$ and $c_2 \neq c_4$ and $c_3 \neq c_4$ and $c_1 \neq c_5$ and $c_1 \neq c_6$ and $c_5 \neq c_6$ and $c_5 \neq c_7$ and $c_6 \neq c_7$ and c_1, c_4 and c_7 are not collinear and c_1, c_4 and c_2 are collinear and c_1, c_4 and c_3 are collinear and c_1, c_7 and c_5 are collinear and c_1, c_7 and c_6 are collinear and c_4, c_5 and c_8 are collinear and c_7, c_2 and c_8 are collinear and c_4, c_6 and c_9 are collinear and c_3, c_7 and c_9 are collinear and c_2, c_6 and c_{10} are collinear and c_3, c_5 and c_{10} are collinear.

Then it is not true that c_4, c_2 and c_7 are collinear or c_4, c_3 and c_7 are collinear or c_2, c_3 and c_7 are collinear or c_4, c_2 and c_5 are collinear or c_4, c_2 and c_6 are collinear or c_4, c_3 and c_5 are collinear or c_4, c_3 and c_6 are collinear or c_2, c_7 and c_5 are collinear or c_2, c_7 and c_6 are collinear or c_3, c_7 and c_5 are collinear or c_3, c_7 and c_6 are collinear or c_2, c_3 and c_5 are collinear or c_2, c_3 and c_6 are collinear or c_7, c_5 and c_4 are collinear or c_7, c_6.

And c_4 are collinear or c_5, c_6 and c_4 are collinear or c_5, c_6 and c_2 are collinear or c_4, c_5 and c_8 are not collinear or c_4, c_6 and c_9 are not collinear or c_2, c_7 and c_8 are not collinear or c_2, c_6 and c_{10} are not collinear or c_3, c_7 and c_9 are not collinear or c_3, c_5 and c_{10} are not collinear.

(20) $\text{conic}(0,0,0,0,0,0)$ = the carrier of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

(21) Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 form the Pascal configuration.

(22) PAPPUS THEOREM AS A SPECIAL CASE OF PASCAL'S THEOREM:
Suppose $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear.

And o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

PROOF: p_1, p_2 and p_3 are collinear. Consider u_1, u_2, u_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that p_1 = the direction of u_1 and p_2 = the direction of u_2 and p_3 = the direction of u_3 and u_1 is not zero and u_2 is not zero and u_3 is not zero and u_1, u_2 and u_3 are lineary dependent. Set $x_1 = (u_2)_{\mathbf{2}} \cdot ((u_3)_{\mathbf{3}}) - (u_2)_{\mathbf{3}} \cdot ((u_3)_{\mathbf{2}})$. Set $x_2 = (u_2)_{\mathbf{3}} \cdot ((u_3)_{\mathbf{1}}) - (u_2)_{\mathbf{1}} \cdot ((u_3)_{\mathbf{3}})$. Set $x_3 = (u_2)_{\mathbf{1}} \cdot ((u_3)_{\mathbf{2}}) - (u_2)_{\mathbf{2}} \cdot ((u_3)_{\mathbf{1}})$. q_1, q_2 and q_3 are collinear.

Consider v_1, v_2, v_3 being elements of $\mathcal{E}^3_{\mathrm{T}}$ such that q_1 = the direction of v_1 and q_2 = the direction of v_2 and q_3 = the direction of v_3 and v_1 is not zero and v_2 is not zero and v_3 is not zero and v_1, v_2 and v_3 are lineary dependent. Set $y_1 = (v_2)_{\mathbf{2}} \cdot ((v_3)_{\mathbf{3}}) - (v_2)_{\mathbf{3}} \cdot ((v_3)_{\mathbf{2}})$. Set $y_2 = (v_2)_{\mathbf{3}} \cdot ((v_3)_{\mathbf{1}}) - (v_2)_{\mathbf{1}} \cdot ((v_3)_{\mathbf{3}})$. Set $y_3 = (v_2)_{\mathbf{1}} \cdot ((v_3)_{\mathbf{2}}) - (v_2)_{\mathbf{2}} \cdot ((v_3)_{\mathbf{1}})$. Set $x_4 = x_1 \cdot y_1$. Set $x_5 = x_2 \cdot y_2$. Set $x_6 = x_3 \cdot y_3$. Set $x_7 = x_1 \cdot y_2 + x_2 \cdot y_1$. Set $x_8 = x_1 \cdot y_3 + x_3 \cdot y_1$. Set $x_1 = x_2 \cdot y_3 + x_3 \cdot y_2$. For every point u of $\mathcal{E}^3_{\mathrm{T}}$, $\text{qfconic}(x_4, x_5, x_6, x_7, x_8, x_1, u) = |(u, u_2 \times u_3)| \cdot |(u, v_2 \times v_3)|$. □

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