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Functional Space Consisted by Continuous Functions on Topological Space

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Summary. In this article, using the Mizar system [1], [2], first we give a definition of a functional space which is constructed from all continuous functions defined on a compact topological space [5]. We prove that this functional space is a Banach space [3]. Next, we give a definition of a function space which is constructed from all continuous functions with bounded support. We also prove that this function space is a normed space.

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1. Real Vector Space of Continuous Functions

From now on S denotes a non empty topological space, T denotes a linear topological space, and X denotes a non empty subset of the carrier of S.

Now we state the propositions:

(1) Let us consider a non empty topological space X, a non empty linear topological space S, functions f, g from X into S, and a point x of X. Suppose f is continuous at x and g is continuous at x. Then $f+g$ is continuous at x.
PROOF: For every neighbourhood G of $(f+g)(x)$, there exists a neighbourhood H of x such that $(f+g)^\circ H \subseteq G$. $\square$

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(2) Let us consider a non empty topological space X, a non empty linear topological space S, a function f from X into S, a point x of X, and a real number a. If f is continuous at x, then $a \cdot f$ is continuous at x.
PROOF: For every neighbourhood G of $(a \cdot f)(x)$, there exists a neighbourhood H of x such that $(a \cdot f)^{\circ} H \subseteq G$. $\square$

(3) Let us consider a non empty topological space X, a non empty linear topological space S, and functions f, g from X into S. If f is continuous and g is continuous, then $f + g$ is continuous.
PROOF: For every point x of X, $f + g$ is continuous at x. $\square$

(4) Let us consider a non empty topological space X, a non empty linear topological space S, a function f from X into S, and a real number a. If f is continuous, then $a \cdot f$ is continuous. The theorem is a consequence of (2).

Let S be a non empty topological space and T be a non empty linear topological space. The continuous functions of S and T yielding a subset of RealVectSpace((the carrier of S), T) is defined by the term

(Def. 1) $\{f$, where f is a function from the carrier of S into the carrier of $T : f$ is continuous$\}$.

Let us observe that the continuous functions of S and T is non empty and functional.

Let us consider a non empty topological space S and a non empty linear topological space T. Now we state the propositions:

(5) The continuous functions of S and T is linearly closed.
PROOF: Set $W =$ the continuous functions of S and T. For every vectors v, u of RealVectSpace((the carrier of S), T) such that v, $u \in$ the continuous functions of S and T holds $v+u \in$ the continuous functions of S and T. For every real number a and for every vector v of RealVectSpace((the carrier of S), T) such that $v \in W$ holds $a \cdot v \in W$. $\square$

(6) $\langle$the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))$\rangle$ is a subspace of RealVectSpace((the carrier of S), T).

Let S be a non empty topological space and T be a non empty linear topological space.

One can verify that $\langle$the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The $\mathbb{R}$-vector space of continuous functions of S and T yielding a strict real linear space is defined by the term

(Def. 2) $\langle$the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))$\rangle$.

Observe that the $\mathbb{R}$-vector space of continuous functions of S and T is constituted functions. Let f be a vector of the $\mathbb{R}$-vector space of continuous functions of S and T and v be an element of S. Let us note that the functor $f(v)$ yields a vector of T. Now we state the propositions:

(7) Let us consider a non empty topological space S, a non empty linear topological space T, and vectors f, g, h of the $\mathbb{R}$-vector space of continuous functions of S and T. Then $h = f + g$ if and only if for every element x of S, $h(x) = f(x) + g(x)$. The theorem is a consequence of (5).

(8) Let us consider a non empty topological space S, a non empty linear topological space T, vectors f, h of the $\mathbb{R}$-vector space of continuous functions of S and T, and a real number a. Then $h = a \cdot f$ if and only if for every element x of S, $h(x) = a \cdot f(x)$. The theorem is a consequence of (5).

(9) Let us consider a non empty topological space S, and a non empty linear topological space T. Then $0_\alpha =$ (the carrier of S) $\longmapsto 0_T$, where α is the $\mathbb{R}$-vector space of continuous functions of S and T. The theorem is a consequence of (5).

Let S be a non empty topological space and T be a non empty linear topological space. Let us note that the carrier of the $\mathbb{R}$-vector space of continuous functions of S and T is functional.

2. Real Vector Space of Continuous Functions (Norm Space Version)

In the sequel S, T denote real normed spaces and X denotes a non empty subset of the carrier of S.

Now we state the proposition:

(10) Let us consider a point x of T. Then (the carrier of S) $\longmapsto x$ is continuous on the carrier of S.

Let S, T be real normed spaces. The continuous functions of S and T yielding a subset of RealVectSpace((the carrier of S), T) is defined by the term

(Def. 3) $\{f$, where f is a function from the carrier of S into the carrier of T : f is continuous on the carrier of $S\}$.

One can check that the continuous functions of S and T is non empty and functional.

Let us consider real normed spaces S, T. Now we state the propositions:

(11) The continuous functions of S and T is linearly closed.
PROOF: Set $W =$ the continuous functions of S and T. For every vectors v, u of RealVectSpace((the carrier of S), T) such that v, $u \in$ the continuous functions of S and T holds $v+u \in$ the continuous functions of S and T. For every real number a and for every vector v of RealVectSpace((the carrier of S), T) such that $v \in W$ holds $a \cdot v \in W$ by [4, (27)]. □

(12) ⟨the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))⟩ is a subspace of RealVectSpace((the carrier of S), T).

Let S, T be real normed spaces. Observe that ⟨the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))⟩ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The $\mathbb{R}$-vector space of continuous functions of S and T yielding a strict real linear space is defined by the term

(Def. 4) ⟨the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T))⟩.

Note that the $\mathbb{R}$-vector space of continuous functions of S and T is constituted functions.

Let f be a vector of the $\mathbb{R}$-vector space of continuous functions of S and T and v be an element of S. One can check that the functor $f(v)$ yields a vector of T. Now we state the propositions:

(13) Let us consider real normed spaces S, T, and vectors f, g, h of the $\mathbb{R}$-vector space of continuous functions of S and T. Then $h = f + g$ if and only if for every element x of S, $h(x) = f(x) + g(x)$. The theorem is a consequence of (11).

(14) Let us consider real normed spaces S, T, vectors f, h of the $\mathbb{R}$-vector space of continuous functions of S and T, and a real number a. Then

$h = a \cdot f$ if and only if for every element x of S, $h(x) = a \cdot f(x)$. The theorem is a consequence of (11).

Let us consider real normed spaces S, T. Now we state the propositions:

(15) The $\mathbb{R}$-vector space of continuous functions of S and T is a subspace of RealVectSpace((the carrier of S), T).

(16) $0_\alpha =$ (the carrier of S) $\longmapsto 0_T$, where α is the $\mathbb{R}$-vector space of continuous functions of S and T. The theorem is a consequence of (11).

Let S, T be real normed spaces and f be an object. Assume $f \in$ the continuous functions of S and T. The functor PartFuncs(f, S, T) yielding a function from S into T is defined by

(Def. 5) *it* $= f$ and *it* is continuous on the carrier of S.

3. Normed Topological Linear Space

We consider normed real linear topological structures which extend real linear topological structures and normed structures and are systems

$$\langle \text{a carrier, a zero, an addition, an external multiplication,}$$
$$\text{a topology, a norm} \rangle$$

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $\mathbb{R} \times$ (the carrier) into the carrier, the topology is a family of subsets of the carrier, the norm is a function from the carrier into $\mathbb{R}$.

Let X be a non empty set, O be an element of X, F be a binary operation on X, G be a function from $\mathbb{R} \times X$ into X, T be a family of subsets of X, and N be a function from X into $\mathbb{R}$. Observe that $\langle X, O, F, G, T, N \rangle$ is non empty and there exists a normed real linear topological structure which is strict and non empty.

Let X be a non empty normed real linear topological structure. We say that X is normed structure if and only if

(Def. 6) there exists a real normed space R such that $R =$ the normed structure of X and the topology of $X =$ the topology of TopSpaceNorm R.

One can verify that there exists a non empty normed real linear topological structure which is strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, real normed space-like, normed structure, and T_2.

A normed linear topological space is a strict, add-continuous, mult-continuous, topological space-like, Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, real normed space-like, normed structure, T_2, non empty normed real linear topological structure. Now we state the propositions:

(17) Every normed linear topological space is a linear topological space.

(18) Every normed linear topological space is a real normed space.

(19) Let us consider a normed linear topological space X, and a real normed space R. Suppose $R =$ the normed structure of X. Let us consider points x, y of X, points x_1, y_1 of R, and a real number a. Suppose $x_1 = x$ and $y_1 = y$. Then

   (i) $x + y = x_1 + y_1$, and

   (ii) $a \cdot x = a \cdot x_1$, and

   (iii) $x - y = x_1 - y_1$, and

   (iv) $\|x\| = \|x_1\|$.

Let us consider a normed linear topological space X, a sequence S of X, and a point x of X. Now we state the propositions:

(20) S is convergent to x if and only if for every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (19).

(21) S is convergent and $x = \lim S$ if and only if for every real number r such that $0 < r$ there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\|S(n) - x\| < r$. The theorem is a consequence of (20).

(22) Let us consider a normed linear topological space X, and a sequence S of X. Suppose S is convergent. Let us consider a real number r. Suppose $0 < r$. Then there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $\|S(n) - \lim S\| < r$. The theorem is a consequence of (20).

(23) Let us consider a normed linear topological space X, and a subset V of X. Then V is open if and only if for every point x of X such that $x \in V$ there exists a real number r such that $r > 0$ and $\{y$, where y is a point of $X : \|x - y\| < r\} \subseteq V$. The theorem is a consequence of (19).

Let us consider a normed linear topological space X, a point x of X, a real number r, and a subset V of X. Now we state the propositions:

(24) If $V = \{y$, where y is a point of $X : \|x - y\| < r\}$, then V is open. The theorem is a consequence of (19).

(25) Suppose $V = \{y$, where y is a point of $X : \|x - y\| \leqslant r\}$. Then V is closed. The theorem is a consequence of (19).

Now we state the propositions:

(26) Let us consider a normed linear topological space X, a real normed space R, a sequence t of X, and a sequence s of R. Suppose $R =$ the normed structure of X and $t = s$ and t is convergent. Then

  (i) s is convergent, and

  (ii) $\lim s = \lim t$.

The theorem is a consequence of (22) and (19).

(27) Let us consider a normed linear topological space X, a real normed space R, a sequence s of X, and a sequence t of R. Suppose $R =$ the normed structure of X and $s = t$. Then s is convergent if and only if t is convergent. The theorem is a consequence of (26), (19), and (21).

(28) Let us consider a normed linear topological space X, and a subset V of X. Then V is closed if and only if for every sequence s_1 of X such that $\operatorname{rng} s_1 \subseteq V$ and s_1 is convergent holds $\lim s_1 \in V$. The theorem is a consequence of (26) and (27).

(29) Let us consider a normed linear topological space X, a real normed space R, a subset V of X, and a subset W of R. Suppose $R =$ the normed structure of X and the topology of $X =$ the topology of TopSpaceNorm R and $V = W$. Then V is closed if and only if W is closed. The theorem is a consequence of (27), (26), and (28).

(30) Let us consider a normed linear topological space X, a subset V of X, and a point x of X. Then V is a neighbourhood of x if and only if there exists a real number r such that $r > 0$ and $\{y$, where y is a point of $X : \|y - x\| < r\} \subseteq V$. The theorem is a consequence of (23) and (24).

(31) Let us consider a normed linear topological space X, and a subset V of X. Then V is compact if and only if for every sequence s_1 of X such that $\operatorname{rng} s_1 \subseteq V$ there exists a sequence s_2 of X such that s_2 is subsequence of s_1 and convergent and $\lim s_2 \in V$. The theorem is a consequence of (27) and (26).

(32) Let us consider a normed linear topological space X, a real normed space R, a subset V of X, and a subset W of R. Suppose $R =$ the normed structure of X and the topology of $X =$ the topology of TopSpaceNorm R and $V = W$. Then V is compact if and only if W is compact. The theorem is a consequence of (31), (26), and (27).

4. Real Norm Space of Continuous Functions

Now we state the propositions:

(33) Let us consider sets X, X_1, a real normed space S, and a partial function f from S to $\mathbb{R}$. Suppose f is continuous on X and $X_1 \subseteq X$. Then f is continuous on X_1.
PROOF: $f{\restriction}X_1$ is continuous in r. □

(34) Let us consider a non empty, compact topological space S, a normed linear topological space T, and a set x. Suppose $x \in$ the continuous functions of S and T. Then $x \in \mathrm{BdFuncs}((\text{the carrier of } S), T)$.

(35) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Then the $\mathbb{R}$-vector space of continuous functions of S and T is a subspace of the set of bounded real sequences from the carrier of S into T. The theorem is a consequence of (34) and (5).

Let S be a non empty, compact topological space and T be a normed linear topological space. The continuous functions norm of S and T yielding a function from the continuous functions of S and T into $\mathbb{R}$ is defined by the term

(Def. 7) $\mathrm{BdFuncsNorm}((\text{the carrier of } S), T){\restriction}(\text{the continuous functions of } S \text{ and } T)$.

The $\mathbb{R}$-norm space of continuous functions of S and T yielding a strict normed structure is defined by the term

(Def. 8) $\langle$the continuous functions of S and T, Zero(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Add(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), Mult(the continuous functions of S and T, RealVectSpace((the carrier of S), T)), the continuous functions norm of S and $T\rangle$.

One can check that the $\mathbb{R}$-norm space of continuous functions of S and T is non empty.

Now we state the propositions:

(36) Let us consider a non empty, compact topological space S, a normed linear topological space T, a point x of the $\mathbb{R}$-norm space of continuous functions of S and T, and a point y of the real normed space of bounded functions from the carrier of S into T. If $x = y$, then $\|x\| = \|y\|$.

(37) Let us consider a non empty, compact topological space S, a normed linear topological space T, a point f of the $\mathbb{R}$-norm space of continuous functions of S and T, and a function g from S into T. Suppose $f = g$. Let us consider a point t of S. Then $\|g(t)\| \leqslant \|f\|$. The theorem is a consequence of (34).

(38) Let us consider a non empty, compact topological space S, a normed linear topological space T, points x_1, x_2 of the $\mathbb{R}$-norm space of continuous functions of S and T, and points y_1, y_2 of the real normed space of bounded functions from the carrier of S into T. If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 + x_2 = y_1 + y_2$. The theorem is a consequence of (5).

(39) Let us consider a non empty, compact topological space S, a normed linear topological space T, a real number a, a point x of the $\mathbb{R}$-norm space of continuous functions of S and T, and a point y of the real normed space of bounded functions from the carrier of S into T. If $x = y$, then $a \cdot x = a \cdot y$. The theorem is a consequence of (5).

Let S be a non empty, compact topological space and T be a normed linear topological space. One can verify that the $\mathbb{R}$-norm space of continuous functions of S and T is non empty, right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

Let us consider a non empty, compact topological space S and a normed linear topological space T. Now we state the propositions:

(40) (The carrier of S) $\longmapsto 0_T = 0_\alpha$, where α is the $\mathbb{R}$-norm space of continuous functions of S and T. The theorem is a consequence of (9).

(41) $0_\alpha = 0_\beta$, where α is the $\mathbb{R}$-norm space of continuous functions of S and T and β is the real normed space of bounded functions from the carrier of S into T. The theorem is a consequence of (40).

Let us consider a non empty, compact topological space S, a normed linear topological space T, and a point F of the $\mathbb{R}$-norm space of continuous functions of S and T. Now we state the propositions:

(42) $0 \leqslant \|F\|$. The theorem is a consequence of (34).

(43) If $F = 0_\alpha$, then $0 = \|F\|$, where α is the $\mathbb{R}$-norm space of continuous functions of S and T. The theorem is a consequence of (34) and (40).

(44) Let us consider a non empty, compact topological space S, a normed linear topological space T, points F, G, H of the $\mathbb{R}$-norm space of continuous functions of S and T, and functions f, g, h from S into T. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of S, $h(x) = f(x) + g(x)$. The theorem is a consequence of (7).

(45) Let us consider a real number a, a non empty, compact topological space S, a normed linear topological space T, points F, G of the $\mathbb{R}$-norm space of continuous functions of S and T, and functions f, g from S into T. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of S, $g(x) = a \cdot f(x)$. The theorem is a consequence of (8).

(46) Let us consider a real number a, a non empty, compact topological space

S, a normed linear topological space T, and points F, G of the $\mathbb{R}$-norm space of continuous functions of S and T. Then

(i) $\|F\| = 0$ iff $F = 0_\alpha$, and

(ii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iii) $\|F + G\| \leqslant \|F\| + \|G\|$,

where α is the $\mathbb{R}$-norm space of continuous functions of S and T. The theorem is a consequence of (34), (38), (36), (41), and (39).

Let S be a non empty, compact topological space and T be a normed linear topological space. Let us observe that the $\mathbb{R}$-norm space of continuous functions of S and T is reflexive, discernible, and real normed space-like.

Now we state the propositions:

(47) Let us consider a non empty, compact topological space S, a normed linear topological space T, points x_1, x_2 of the $\mathbb{R}$-norm space of continuous functions of S and T, and points y_1, y_2 of the real normed space of bounded functions from the carrier of S into T. If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 - x_2 = y_1 - y_2$. The theorem is a consequence of (39) and (38).

(48) Let us consider a non empty, compact topological space S, a normed linear topological space T, points F, G, H of the $\mathbb{R}$-norm space of continuous functions of S and T, and functions f, g, h from S into T. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of S, $h(x) = f(x) - g(x)$. The theorem is a consequence of (44).

(49) Let us consider a non empty topological space S, a normed linear topological space T, a sequence H of partial functions from the carrier of S into the carrier of T, and a function L_1 from S into T. Suppose H is uniform-convergent on the carrier of S and for every natural number n, there exists a function H_1 from S into T such that $H_1 = H(n)$ and H_1 is continuous and $L_1 = \lim_\alpha H$. Then L_1 is continuous, where α is the carrier of S.

PROOF: For every point x of S, L_1 is continuous at x by (30), [7, (33),(11)]. □

(50) Let us consider a non empty, compact topological space S, a normed linear topological space T, and a subset Y of the carrier of the real normed space of bounded functions from the carrier of S into T. Suppose $Y =$ the continuous functions of S and T. Then Y is closed. The theorem is a consequence of (49).

(51) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Let us consider a sequence s_3 of the $\mathbb{R}$-norm space of continuous functions of S and T.

If s_3 is Cauchy sequence by norm, then s_3 is convergent. The theorem is a consequence of (34), (47), (36), and (50).

(52) Let us consider a non empty, compact topological space S, and a normed linear topological space T. Suppose T is complete. Then the $\mathbb{R}$-norm space of continuous functions of S and T is complete. The theorem is a consequence of (51).

5. Some Properties of Support

Let X be a zero structure and f be a (the carrier of X)-valued function. The functor support f yielding a set is defined by

(Def. 9) for every object x, $x \in it$ iff $x \in \operatorname{dom} f$ and $f_{/x} \neq 0_X$.

Now we state the proposition:

(53) Let us consider a zero structure X, and a (the carrier of X)-valued function f. Then support $f \subseteq \operatorname{dom} f$.

Let X be a non empty topological space, T be a real linear space, and f be a function from X into T. One can verify that the functor support f yields a subset of X. Now we state the propositions:

(54) Let us consider a non empty topological space X, a real linear space T, and functions f, g from X into T. Then $\operatorname{support}(f+g) \subseteq \operatorname{support} f \cup \operatorname{support} g$.

(55) Let us consider a non empty topological space X, a real linear space T, a function f from X into T, and a real number a. Then $\operatorname{support}(a \cdot f) \subseteq \operatorname{support} f$.

6. Space of Real-valued Continuous Functionals with Bounded Support

Let X be a non empty topological space and T be a normed linear topological space. The functor $\mathrm{C}_0\mathrm{Functions}(X, T)$ yielding a non empty subset of $\mathrm{RealVectSpace}((\text{the carrier of } X), T)$ is defined by the term

(Def. 10) $\{f$, where f is a function from the carrier of X into the carrier of T : f is continuous and there exists a non empty subset Y of X such that Y is compact and $\overline{\operatorname{support} f} \subseteq Y\}$.

Now we state the propositions:

(56) Let us consider a non empty topological space X, a normed linear topological space T, and elements v, u of $\mathrm{RealVectSpace}((\text{the carrier of } X), T)$.

Suppose v, $u \in \mathrm{C_0Functions}(X, T)$. Then $v + u \in \mathrm{C_0Functions}(X, T)$. The theorem is a consequence of (5) and (54).

(57) Let us consider a non empty topological space X, a normed linear topological space T, a real number a, and an element u of RealVectSpace((the carrier of X), T). Suppose $u \in \mathrm{C_0Functions}(X, T)$. Then $a \cdot u \in \mathrm{C_0Functions}(X, T)$. The theorem is a consequence of (5) and (55).

(58) Let us consider a non empty topological space X, and a normed linear topological space T. Then $\mathrm{C_0Functions}(X, T)$ is linearly closed.

Let X be a non empty topological space and T be a normed linear topological space. Let us note that $\mathrm{C_0Functions}(X, T)$ is non empty and linearly closed.

The functor $\mathrm{RV_{SP}C_0Functions}(X, T)$ yielding a real linear space is defined by the term

(Def. 11) $\langle \mathrm{C_0Functions}(X, T), \mathrm{Zero}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)), \mathrm{Add}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)), \mathrm{Mult}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)) \rangle$.

Now we state the propositions:

(59) Let us consider a non empty topological space X, and a normed linear topological space T. Then $\mathrm{RV_{SP}C_0Functions}(X, T)$ is a subspace of RealVectSpace((the carrier of X), T).

(60) Let us consider a non empty topological space X, a normed linear topological space T, and a set x. Suppose $x \in \mathrm{C_0Functions}(X, T)$. Then $x \in$ BdFuncs((the carrier of X), T).

PROOF: Consider f being a function from the carrier of X into the carrier of T such that $f = x$ and f is continuous and there exists a non empty subset Y of X such that Y is compact and $\overline{\mathrm{support}\, f} \subseteq Y$. Consider Y being a non empty subset of X such that Y is compact and $\overline{\mathrm{support}\, f} \subseteq Y$. Consider K being a real number such that $0 \leqslant K$ and for every point x of X such that $x \in Y$ holds $\|f(x)\| \leqslant K$. For every element x of X, $\|f(x)\| \leqslant K$. $\square$

Let X be a non empty topological space and T be a normed linear topological space. The functor $\mathrm{NormC_0Functions}(X, T)$ yielding a function from $\mathrm{C_0Functions}(X, T)$ into $\mathbb{R}$ is defined by the term

(Def. 12) BdFuncsNorm((the carrier of X), T) $\restriction \mathrm{C_0Functions}(X, T)$.

The functor $\mathrm{NormSp_{C_0}Functions}(X, T)$ yielding a normed structure is defined by the term

(Def. 13) $\langle \mathrm{C_0Functions}(X, T), \mathrm{Zero}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)), \mathrm{Add}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)), \mathrm{Mult}(\mathrm{C_0Functions}(X, T), \mathrm{RealVectSpace}((\text{the carrier of } X), T)),$

NormC$_0$Functions$(X, T)\rangle$.

Let us note that NormSp$_{\mathrm{C}_0}$Functions(X, T) is strict and non empty.

Now we state the proposition:

(61) Let us consider a non empty topological space X, a normed linear topological space T, and a set x. Suppose $x \in$ C$_0$Functions(X, T). Then $x \in$ the continuous functions of X and T.

Let us consider a non empty topological space X and a normed linear topological space T. Now we state the propositions:

(62) $0_{\mathrm{RV_{SP}C_0Functions}(X,T)} = X \longmapsto 0_T$.

(63) $0_{\mathrm{NormSp_{C_0}Functions}(X,T)} = X \longmapsto 0_T$. The theorem is a consequence of (62).

(64) Let us consider a non empty topological space X, a normed linear topological space T, points x_1, x_2 of NormSp$_{\mathrm{C}_0}$Functions(X, T), and points y_1, y_2 of the real normed space of bounded functions from the carrier of X into T. If $x_1 = y_1$ and $x_2 = y_2$, then $x_1 + x_2 = y_1 + y_2$.

(65) Let us consider a non empty topological space X, a normed linear topological space T, a real number a, a point x of NormSp$_{\mathrm{C}_0}$Functions(X, T), and a point y of the real normed space of bounded functions from the carrier of X into T. If $x = y$, then $a \cdot x = a \cdot y$.

(66) Let us consider a real number a, a non empty topological space X, a normed linear topological space T, and points F, G of NormSp$_{\mathrm{C}_0}$Functions(X, T). Then

(i) $\|F\| = 0$ iff $F = 0_{\mathrm{NormSp_{C_0}Functions}(X,T)}$, and

(ii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iii) $\|F + G\| \leqslant \|F\| + \|G\|$.

PROOF: $\|F\| = 0$ iff $F = 0_{\mathrm{NormSp_{C_0}Functions}(X,T)}$. $\|a \cdot F\| = |a| \cdot \|F\|$. $\|F + G\| \leqslant \|F\| + \|G\|$ by (60), (64) [6, (21)]. □

(67) Let us consider a non empty topological space X, and a normed linear topological space T. Then NormSp$_{\mathrm{C}_0}$Functions(X, T) is real normed space-like.

Let X be a non empty topological space and T be a normed linear topological space. Let us note that NormSp$_{\mathrm{C}_0}$Functions(X, T) is reflexive, discernible, real normed space-like, vector distributive.

And let us observe that NormSp$_{\mathrm{C}_0}$Functions(X, T) is scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(68) Let us consider a non empty topological space X, and a normed linear topological space T. Then $\mathrm{NormSp}_{\mathrm{C}_0}\mathrm{Functions}(X, T)$ is a real normed space.

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