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Miscellaneous Graph Preliminaries. Part I

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Summary. This article contains many auxiliary theorems which were missing in the Mizar Mathematical Library to the best of the author's knowledge. Most of them regard graph theory as formalized in the GLIB series and are needed in upcoming articles.

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0. Introduction

A generalized approach to graph theory as it was done in [2, 4] in contrast to [9, 3] is rather uncommon. To avoid duplication of the same theorems in different formalization frameworks in the Mizar Mathematical Library [1], a generalized approach to formalization is preferable (cf. [8, 7]). However, due to the sheer amount of "obvious facts" such an approach brings with it, it is only natural some of them not immediately needed slip the initial formalization process. This article, like its precedessor [5], aims to fill some of the gaps that emerged.

Many theorems in this article regard the property of a walk in a graph to be the shortest one, which have been rather neglected in the author's work on graphs in Mizar until now. Another good portion is concered with theorems about graph mappings which are missing from [7]. Further worthy of note is the theorem that combines adding an edge or adjacent vertex with the reversal of

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the edge to be added and the two theorems noting that a connected graph is unicyclic if and only if the connected subgraph it can be constructed from by adding an edge is a tree.

1. Preliminaries not Directly Related to Graphs

Now we state the propositions:

(1) Let us consider sets X_1, X_2, X_3, X_4, X_5, X_6, X_7. Then it is not true that $X_1 \in X_2$ and $X_2 \in X_3$ and $X_3 \in X_4$ and $X_4 \in X_5$ and $X_5 \in X_6$ and $X_6 \in X_7$ and $X_7 \in X_1$.

(2) Let us consider sets X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8. Then it is not true that $X_1 \in X_2$ and $X_2 \in X_3$ and $X_3 \in X_4$ and $X_4 \in X_5$ and $X_5 \in X_6$ and $X_6 \in X_7$ and $X_7 \in X_8$ and $X_8 \in X_1$.

One can verify that every function which is one-to-one and constant is also trivial. Now we state the proposition:

(3) Let us consider a function f. Then f is non empty and constant if and only if there exists an object y such that $\operatorname{rng} f = \{y\}$.

Let X be a set. Observe that there exists a many sorted set indexed by X which is one-to-one and there exists an X-defined function which is total and one-to-one.

Let X be a non empty set. One can check that there exists an X-defined function which is total, one-to-one, and non empty.

The scheme *LambdaDf* deals with non empty sets $\mathcal{C}$, $\mathcal{D}$ and a unary functor $\mathcal{F}$ yielding an object and states that

(Sch. 1) There exists a function f from $\mathcal{C}$ into $\mathcal{D}$ such that for every element x of $\mathcal{C}$, $f(x) = \mathcal{F}(x)$

provided

- for every element x of $\mathcal{C}$, $\mathcal{F}(x) \in \mathcal{D}$.

Now we state the proposition:

(4) Let us consider a one-to-one function f, and an object y. Suppose $\operatorname{rng} f = \{y\}$. Then there exists an object x such that $f = x \dot{\longmapsto} y$.

Let f be a one-to-one function. Note that $f^{\smile}$ is one-to-one. Let f be a function and g be a one-to-one function. Let us observe that $\langle f, g \rangle$ is one-to-one and $\langle g, f \rangle$ is one-to-one. Now we state the propositions:

(5) Let us consider an empty function f. Then ${}^{\circ}f = \emptyset \dot{\longmapsto} \emptyset$.

Let f be a one-to-one function. One can check that ${}^{\circ}f$ is one-to-one.

(6) Let us consider a non empty, one-to-one function f, and a non empty subset X of $2^{\mathrm{dom}\, f}$. Then $\mathrm{rng}(^\circ f{\restriction}X) =$ the set of all $f^\circ x$ where x is an element of X.

(7) Let us consider a function f, and one-to-one functions g, h. Suppose $h = f{+}{\cdot}g$. Then $h^{-1}{\restriction}\mathrm{rng}\, g = g^{-1}$.

(8) Let us consider functions f, g, h. If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, h$, then $(g{+}{\cdot}h)\cdot f = h \cdot f$.

(9) Let us consider a function f, and a one-to-one function g. Then $(f{+}{\cdot}g)\cdot(g^{-1}) = \mathrm{id}_{\mathrm{rng}\, g}$. The theorem is a consequence of (8).

Observe that every binary relation which is reflexive and connected is also strongly connected. Now we state the propositions:

(10) Let us consider a set X, and a binary relation R on X. Then R is antisymmetric if and only if $R \setminus (\mathrm{id}_X)$ is asymmetric.

(11) Let us consider a set X. Suppose X is mutually-disjoint. Then $X \setminus \{\emptyset\}$ is a partition of $\bigcup X$.

Let X be a set. Let us note that every partition of X is mutually-disjoint.

(12) Let us consider cardinal numbers M, N, and a function f. Suppose $M \subseteq \overline{\overline{\mathrm{dom}\, f}}$ and for every object x such that $x \in \mathrm{dom}\, f$ holds $N \subseteq \overline{\overline{f(x)}}$. Then $M \cdot N \subseteq \sum \mathrm{Card}\, f$.

(13) Let us consider sets X, x. Suppose $x \in X$. Then $(\mathrm{disjoint}\,\mathrm{Card}\,\mathrm{id}_X)(x) = \overline{\overline{x}} \times \{x\}$.

(14) Let us consider a set X. Suppose X is mutually-disjoint. Then $\sum \mathrm{Card}\,\mathrm{id}_X = \overline{\overline{\bigcup X}}$. The theorem is a consequence of (11) and (13).

(15) Let us consider a set X, and cardinal numbers M, N. Suppose X is mutually-disjoint and $M \subseteq \overline{\overline{X}}$ and for every set Y such that $Y \in X$ holds $N \subseteq \overline{\overline{Y}}$. Then $M \cdot N \subseteq \overline{\overline{\bigcup X}}$. The theorem is a consequence of (12) and (14).

(16) Let us consider a compatible, functional set F. Suppose for every function f_1 such that $f_1 \in F$ holds f_1 is one-to-one and for every function f_2 such that $f_2 \in F$ and $f_1 \neq f_2$ holds $\mathrm{rng}\, f_1$ misses $\mathrm{rng}\, f_2$. Then $\bigcup F$ is one-to-one.

2. Into GLIB_000

Let G be a non trivial graph. Observe that there exists a subset of the vertices of G which is non empty and proper. Now we state the propositions:

(17) Let us consider a graph G, and a set X. Then $G.\mathrm{edgesBetween}(X, X) = G.\mathrm{edgesBetween}(X)$.

(18) Let us consider a graph G. Then G is trivial if and only if the vertices of G is trivial.

(19) Let us consider a graph G_1. Then every subgraph of G_1 is a subgraph of G_1 induced by the vertices of G_2 and the edges of G_2.

(20) Let us consider graphs G_1, G_2, and a spanning subgraph G_3 of G_1. If $G_2 \approx G_3$, then G_2 is a spanning subgraph of G_1.

(21) Let us consider a graph G, and an object e. Suppose $e \in$ the edges of G. Then $e \in G$.edgesBetween($\{$(the source of $G)(e)$, (the target of $G)(e)\}$).

(22) Let us consider a graph G. Then $G \approx$ createGraph(the vertices of G, the edges of G, the source of G, the target of G).

(23) Let us consider a graph G, and a vertex v of G. Then v is endvertex if and only if v.degree() $= 1$.
PROOF: v.inDegree() $= 1$ and v.outDegree() $= 0$ or v.inDegree() $= 0$ and v.outDegree() $= 1$. $\square$

(24) Let us consider a loopless graph G, and a vertex v of G. Then
   (i) v.inNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$, and
   (ii) v.outNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$, and
   (iii) v.allNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$.

(25) Let us consider a graph G. Suppose for every vertex v of G, v.inNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$ or v.outNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$ or v.allNeighbors() $\subseteq$ (the vertices of $G) \setminus \{v\}$. Then G is loopless.

Let X be a set and G be a graph. Let us note that $X \longmapsto G$ is graph-yielding.

Let x be an object. Let us note that $x \dot{\longmapsto} G$ is graph-yielding.

Let X be a set. Let us note that there exists a many sorted set indexed by X which is graph-yielding.

Let X be a non empty set. One can verify that there exists a many sorted set indexed by X which is non empty and graph-yielding.

Let f be a graph-yielding many sorted set indexed by X and x be an element of X. One can verify that the functor $f(x)$ yields a graph.

3. Into GLIB_001

Let G be a graph and P be a path of G. One can verify that P.vertexSeq()$\restriction P$.length() is one-to-one. Now we state the propositions:

(26) Let us consider a graph G, and a path P of G. Then P.length() $\subseteq$ G.order().

(27) Let us consider a graph G, and a trail T of G. Then T.length() $\subseteq G$.size().

(28) Let us consider a graph G, and a walk W of G. Suppose $\operatorname{len} W = 3$ or W.length() $= 1$. Then there exists an object e such that

(i) e joins W.first() and W.last() in G, and

(ii) $W = G$.walkOf(W.first(), e, W.last()).

(29) Let us consider a graph G, a walk W of G, and an object e. Suppose $e \in W$.edges() and $e \notin G$.loops() and W is circuit-like. Then there exists an object e_0 such that

(i) $e_0 \in W$.edges(), and

(ii) $e_0 \neq e$.

Proof: Consider n being an odd element of $\mathbb{N}$ such that $n < \operatorname{len} W$ and $W(n+1) = e$. $\operatorname{len} W > 3$. □

(30) Let us consider a graph G, a path P of G, and odd elements n, m of $\mathbb{N}$. Suppose $n < m \leqslant \operatorname{len} P$ and ($n \neq 1$ or $m \neq \operatorname{len} P$). Then P.cut(n, m) is open.

(31) Let us consider a graph G, a closed walk W of G, and an odd element n of $\mathbb{N}$. Suppose $n < \operatorname{len} W$. Then

(i) (W.cut($n+2, \operatorname{len} W$)).append(($W$.cut($1, n$))) is a walk from $W(n+2)$ to $W(n)$, and

(ii) if W is trail-like, then (W.cut($n+2, \operatorname{len} W$)).edges() misses ($W$.cut(1, n)).edges() and ((W.cut($n+2, \operatorname{len} W$)).append(($W$.cut($1, n$)))).edges () $= W$.edges() $\setminus \{W(n+1)\}$, and

(iii) if W is path-like, then (W.cut($n+2, \operatorname{len} W$)).vertices()$\cap$($W$.cut($1, n$)) .vertices() $= \{W$.first()$\}$ and if $W(n+1) \notin G$.loops(), then (W.cut($n+$ $2, \operatorname{len} W$)).append((W.cut($1, n$))) is open and (W.cut($n+2, \operatorname{len} W$)) .append(($W$.cut($1, n$))) is path-like.

Proof: Set $W_7 = W$.cut($n+2, \operatorname{len} W$). Set $W_8 = W$.cut($1, n$). Set $W' =$ W_7.append(W_8). If W is trail-like, then W_7.edges() misses W_8.edges() and W'.edges() $= W$.edges() $\setminus \{W(n+1)\}$. If $W(n+1) \notin G$.loops(), then W' is open. □

(32) Let us consider a graph G, a walk W_1 of G, and objects e, x, y. Suppose e joins x and y in G and $e \in W_1$.edges() and W_1 is cycle-like. Then there exists a path W_2 of G such that

(i) W_2 is a walk from x to y, and

(ii) W_2.edges() $= W_1$.edges() $\setminus \{e\}$, and

(iii) if $e \notin G$.loops(), then W_2 is open.

The theorem is a consequence of (31).

(33) Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Then $\mathrm{len}\, W_1 \leqslant \mathrm{len}\, W_2$ if and only if W_1.length() $\leqslant W_2$.length().

(34) Let us consider a graph G, and a walk W of G. Then W.length() = W.reverse().length().

(35) Let us consider a graph G, a walk W of G, and an object e. If $e \notin W$.last().edgesInOut(), then W.addEdge$(e) = W$.

(36) Let us consider a graph G, a walk W of G, and objects e, x. Suppose e joins W.last() and x in G. Then (W.addEdge(e)).length() = W.length() + 1.

(37) Let us consider a graph G_1, a set E, a subgraph G_2 of G_1 with edges E removed, and a walk W_1 of G_1. If W_1.edges() misses E, then W_1 is a walk of G_2.

4. Into GLIB_002

Let us consider graphs G_1, G_2 and a component G_3 of G_1. Now we state the propositions:

(38) If $G_2 \approx G_3$, then G_2 is a component of G_1.

(39) If $G_1 \approx G_2$, then G_3 is a component of G_2.

Now we state the proposition:

(40) Let us consider a tree-like graph G, and a spanning subgraph H of G. If H is connected, then $G \approx H$.

Proof: The edges of $G \subseteq$ the edges of H. □

Let G be a graph. Note that every element of G.componentSet() is non empty and G.componentSet() is mutually-disjoint.

5. Into CHORD

Now we state the propositions:

(41) Let us consider a graph G, and vertices v, w of G. Then v and w are adjacent if and only if $w \in v$.allNeighbors().

(42) Let us consider a graph G, a set S, and a vertex v of G. Suppose $v \notin S$ and S meets G.reachableFrom(v). Then G.adjacentSet$(S) \neq \emptyset$.

Proof: Consider w being an object such that $w \in S$ and $w \in G$.reachable From(v). Consider W being a walk of G such that W is a walk from v to w. There exists an odd natural number n such that $n < \mathrm{len}\, W$ and $W(n) \notin S$ and $W(n+2) \in S$. Consider n being an odd natural number such that $n < \mathrm{len}\, W$ and $W(n) \notin S$ and $W(n+2) \in S$. □

Let G be a non trivial, connected graph and S be a non empty, proper subset of the vertices of G. One can check that G.adjacentSet(S) is non empty.

Now we state the propositions:

(43) Let us consider a complete graph G, and a vertex v of G. Then (the vertices of G) $\setminus \{v\} \subseteq v$.allNeighbors().

(44) Let us consider a loopless, complete graph G, and a vertex v of G. Then v.allNeighbors() = (the vertices of G) $\setminus \{v\}$. The theorem is a consequence of (43).

(45) Let us consider a simple, complete graph G, and a vertex v of G. Then v.degree() + 1 = G.order(). The theorem is a consequence of (44).

Let G be a graph. Observe that every walk of G which is trivial is also minimum length and there exists a walk of G which is minimum length and path-like.

Let W be a minimum length walk of G. One can check that W.reverse() is minimum length.

Now we state the propositions:

(46) Let us consider a graph G_1, a subgraph G_2 of G_1, a walk W_1 of G_1, and a walk W_2 of G_2. If $W_1 = W_2$ and W_1 is minimum length, then W_2 is minimum length.

(47) Let us consider a graph G, a vertex v of G, and a walk W of G. Suppose W is a walk from v to v. Then W is minimum length if and only if $W = G$.walkOf(v).

(48) Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Suppose $G_1 \approx G_2$ and $W_1 = W_2$ and W_1 is minimum length. Then W_2 is minimum length.

6. INTO GLIB_006

Now we state the propositions:

(49) Let us consider graphs G_1, G_2. Suppose the vertices of $G_2 \subseteq$ the vertices of G_1 and for every objects e, x, y such that e joins x to y in G_2 holds e joins x to y in G_1. Then

(i) G_2 is a subgraph of G_1, and

(ii) G_1 is a supergraph of G_2.

(50) Let us consider a graph G_1, a subgraph G_3 of G_1, objects v, e, w, and a supergraph G_2 of G_3 extended by e between vertices v and w. If e joins v to w in G_1, then G_2 is a subgraph of G_1.

(51) Let us consider a tree-like graph G, vertices v_1, v_2 of G, an object e, and a supergraph H of G extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G. Then

(i) H is not acyclic, and

(ii) for every walks W_1, W_2 of H such that W_1 is cycle-like and W_2 is cycle-like holds W_1.edges() $=$ W_2.edges().

PROOF: $e \in W_1$.edges(). $e \in W_2$.edges(). Consider W_3 being a path of H such that W_3 is a walk from v_1 to v_2 and W_3.edges() $=$ W_1.edges() $\setminus \{e\}$ and if $e \notin H$.loops(), then W_3 is open. Consider W_4 being a path of H such that W_4 is a walk from v_1 to v_2 and W_4.edges() $=$ W_2.edges() $\setminus \{e\}$ and if $e \notin H$.loops(), then W_4 is open. □

(52) Let us consider a connected graph G. Suppose there exist vertices v_1, v_2 of G and there exists an object e and there exists a supergraph H of G extended by e between vertices v_1 and v_2 such that $e \notin$ the edges of G and for every walks W_1, W_2 of H such that W_1 is cycle-like and W_2 is cycle-like holds W_1.edges() $=$ W_2.edges(). Then G is tree-like.
PROOF: G is acyclic by [6, (75),(24),(105)], [8, (16)]. □

(53) Let us consider a graph G_2, objects v, e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Then

(i) the vertices of $G_1 \subseteq$ (the vertices of G_2) $\cup \{v, w\}$, and

(ii) the edges of $G_1 \subseteq$ (the edges of G_2) $\cup \{e\}$.

(54) Let us consider a graph G_2, vertices v, v_2 of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1. Suppose $v_1 = v_2$ and $v \notin G_2$.reachableFrom(v_2) and $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2).

(55) Let us consider a graph G_2, vertices w, v_2 of G_2, objects v, e, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1. Suppose $v_1 = v_2$ and $w \notin G_2$.reachableFrom(v_2) and $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then G_1.reachableFrom(v_1) $=$ G_2.reachableFrom(v_2).

(56) Let us consider a graph G_2, a vertex v of G_2, objects e, w, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1. Suppose $v_1 = v$ and $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then G_1.reachableFrom(v_1) $=$ (G_2.reachableFrom(v)) $\cup \{w\}$.

(57) Let us consider a graph G_2, objects v, e, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, and a vertex v_1 of G_1.

Suppose $v_1 = w$ and $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then G_1.reachableFrom$(v_1) = (G_2$.reachableFrom$(w)) \cup \{v\}$.

(58) Let us consider a graph G_2, a vertex v of G_2, objects e, w, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2. Then G_1.componentSet() $=$ G_2.componentSet() $\setminus \{G_2$.reachableFrom$(v)\} \cup \{(G_2$.reachableFrom$(v)) \cup \{w\}\}$. The theorem is a consequence of (54) and (56).

(59) Let us consider a graph G_2, objects v, e, a vertex w of G_2, and a supergraph G_1 of G_2 extended by v, w and e between them. Suppose $e \notin$ the edges of G_2 and $v \notin$ the vertices of G_2. Then G_1.componentSet() $=$ G_2.componentSet() $\setminus \{G_2$.reachableFrom$(w)\} \cup \{(G_2$.reachableFrom$(w)) \cup \{v\}\}$. The theorem is a consequence of (55) and (57).

(60) Let us consider a graph G_2, objects v, e, w, a supergraph G_1 of G_2 extended by v, w and e between them, a walk W_1 of G_1, and a walk W_2 of G_2. If $W_1 = W_2$ and W_2 is minimum length, then W_1 is minimum length. The theorem is a consequence of (48).

(61) Let us consider a non trivial, connected graph G_1, and a non spanning subgraph G_2 of G_1. Then there exist objects v, e, w such that

- (i) $v \neq w$, and
- (ii) e joins v to w in G_1, and
- (iii) $e \notin$ the edges of G_2, and
- (iv) every supergraph of G_2 extended by v, w and e between them is a subgraph of G_1, and
- (v) $v \in$ the vertices of G_2 and $w \notin$ the vertices of G_2 or $v \notin$ the vertices of G_2 and $w \in$ the vertices of G_2.

Proof: Set $S =$ the vertices of G_2. Set $v_0 =$ the element of G_1.adjacentSet (S). Consider w_0 being a vertex of G_1 such that $w_0 \in S$ and v_0 and w_0 are adjacent. Consider e being an object such that e joins v_0 and w_0 in G_1. $e \notin$ the edges of G_2. □

(62) Let us consider a graph G_2, a vertex v of G_2, objects e, w, x, a supergraph G_1 of G_2 extended by v, w and e between them, a walk W_1 of G_1, and a walk W_2 of G_2. Suppose $W_1 = W_2$ and W_2 is minimum length and a walk from x to v and $e \notin$ the edges of G_2. Then W_1.addEdge(e) is minimum length. The theorem is a consequence of (60) and (35).

(63) Let us consider a graph G_2, objects v, e, x, a vertex w of G_2, a supergraph G_1 of G_2 extended by v, w and e between them, a walk W_1 of G_1, and a walk W_2 of G_2. Suppose $W_1 = W_2$ and W_2 is minimum length and a walk

from x to w and $e \notin$ the edges of G_2. Then W_1.addEdge(e) is minimum length. The theorem is a consequence of (60) and (35).

Observe that there exists a graph-yielding function which is non empty, non non-directed-multi, and non non-multi and there exists a graph-yielding function which is non empty, non acyclic, and non connected and there exists a graph-yielding function which is non empty and non edgeless and there exists a graph-yielding function which is non empty and non loopfull.

7. Into GLIB_007

Now we state the propositions:

(64) Let us consider graphs G_2, G_3, sets V, E, a supergraph G_1 of G_3 extended by the vertices from V, and a graph G_4 given by reversing directions of the edges E of G_3. Then G_2 is a graph given by reversing directions of the edges E of G_1 if and only if G_2 is a supergraph of G_4 extended by the vertices from V. The theorem is a consequence of (49).

(65) Let us consider graphs G_2, G_3, objects v, e, w, and a supergraph G_1 of G_3 extended by e between vertices v and w. Suppose $e \notin$ the edges of G_3. Then G_2 is a graph given by reversing directions of the edges $\{e\}$ of G_1 if and only if G_2 is a supergraph of G_3 extended by e between vertices w and v. The theorem is a consequence of (49).

(66) Let us consider graphs G_2, G_3, objects v, e, w, and a supergraph G_1 of G_3 extended by v, w and e between them. Suppose $e \notin$ the edges of G_3. Then G_2 is a graph given by reversing directions of the edges $\{e\}$ of G_1 if and only if G_2 is a supergraph of G_3 extended by w, v and e between them. The theorem is a consequence of (65).

(67) Let us consider a graph G_1, a set E, a graph G_2 given by reversing directions of the edges E of G_1, a walk W_1 of G_1, and a walk W_2 of G_2. If $W_1 = W_2$, then W_1 is minimum length iff W_2 is minimum length.

8. Into GLIB_008

Now we state the proposition:

(68) Let us consider an edgeless graph G_1, and a graph G_2. Then G_1 is a subgraph of G_2 if and only if the vertices of $G_1 \subseteq$ the vertices of G_2.

One can check that there exists a graph which is loopless and non edgeless.

9. Into GLIB_009

Let G be a graph. Note that there exists a subgraph of G which is plain, spanning, and acyclic and there exists a subgraph of G which is plain and tree-like and there exists a component of G which is plain.

Now we state the proposition:

(69) Let us consider a plain graph G. Then $G =$ createGraph(the vertices of G, the edges of G, the source of G, the target of G).

Let us consider a graph G and a subgraph H of G with loops removed. Now we state the propositions:

(70) the edges of $G = G$.loops() if and only if H is edgeless.

(71) Every loopless subgraph of G is a subgraph of H.
Proof: (The edges of H') $\cap\, G$.loops() $= \emptyset$. □

(72) Let us consider a graph G_1, and a subgraph G_2 of G_1 with loops removed. Then every minimum length walk of G_1 is a walk of G_2. The theorem is a consequence of (37).

(73) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, a walk W_1 of G_1, and a walk W_2 of G_2. If $W_1 = W_2$, then W_1 is minimum length iff W_2 is minimum length. The theorem is a consequence of (46), (37), and (47).

(74) Let us consider a graph G_1, a subgraph G_2 of G_1 with loops removed, vertices v_1, w_1 of G_1, and vertices v_2, w_2 of G_2. Suppose $v_1 = v_2$ and $w_1 = w_2$ and $v_1 \neq w_1$. Then v_1 and w_1 are adjacent if and only if v_2 and w_2 are adjacent. The theorem is a consequence of (41).

(75) Let us consider a graph G_1, a subgraph G_2 of G_1 with parallel edges removed, vertices v_1, w_1 of G_1, and vertices v_2, w_2 of G_2. Suppose $v_1 = v_2$ and $w_1 = w_2$. Then v_1 and w_1 are adjacent if and only if v_2 and w_2 are adjacent. The theorem is a consequence of (41).

(76) Let us consider a graph G_1, a subgraph G_2 of G_1 with directed-parallel edges removed, vertices v_1, w_1 of G_1, and vertices v_2, w_2 of G_2. Suppose $v_1 = v_2$ and $w_1 = w_2$. Then v_1 and w_1 are adjacent if and only if v_2 and w_2 are adjacent. The theorem is a consequence of (41).

(77) Let us consider a graph G_1, a simple graph G_2 of G_1, vertices v_1, w_1 of G_1, and vertices v_2, w_2 of G_2. Suppose $v_1 = v_2$ and $w_1 = w_2$ and $v_1 \neq w_1$. Then v_1 and w_1 are adjacent if and only if v_2 and w_2 are adjacent. The theorem is a consequence of (75) and (74).

(78) Let us consider a graph G_1, a directed-simple graph G_2 of G_1, vertices v_1, w_1 of G_1, and vertices v_2, w_2 of G_2. Suppose $v_1 = v_2$ and $w_1 = w_2$

and $v_1 \neq w_1$. Then v_1 and w_1 are adjacent if and only if v_2 and w_2 are adjacent. The theorem is a consequence of (76) and (74).

10. Into GLIB_010

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Now we state the propositions:

(79) If $v_2 = (F_{\mathbb{V}})(v_1)$ and F is total, then $(F_{\mathbb{V}})^{\circ}(G_1.\text{reachableFrom}(v_1)) \subseteq G_2.\text{reachableFrom}(v_2)$.

(80) Suppose $v_1 \in \text{dom}(F_{\mathbb{V}})$ and $v_2 = (F_{\mathbb{V}})(v_1)$ and F is one-to-one and onto. Then $G_2.\text{reachableFrom}(v_2) \subseteq (F_{\mathbb{V}})^{\circ}(G_1.\text{reachableFrom}(v_1))$.

(81) If $v_2 = (F_{\mathbb{V}})(v_1)$ and F is isomorphism, then $(F_{\mathbb{V}})^{\circ}(G_1.\text{reachableFrom}(v_1)) = G_2.\text{reachableFrom}(v_2)$. The theorem is a consequence of (79) and (80).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(82) Suppose F is isomorphism. Then $G_2.\text{componentSet}() =$ the set of all $(F_{\mathbb{V}})^{\circ}C$ where C is an element of $G_1.\text{componentSet}()$. The theorem is a consequence of (81).

(83) If F is isomorphism, then $G_1.\text{numComponents}() = G_2.\text{numComponents}()$. The theorem is a consequence of (6) and (82).

Let G be a loopless graph. Let us note that every graph which is G-isomorphic is also loopless. Now we state the proposition:

(84) Let us consider graphs G_1, G_2, G_3, G_4, an empty partial graph mapping F_1 from G_1 to G_2, and an empty partial graph mapping F_2 from G_3 to G_4. Then $F_1 = F_2$.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(85) (i) $F \restriction \text{dom}\, F = F$, and

(ii) $\text{rng}\, F \upharpoonleft F = F$.

The theorem is a consequence of (84).

(86) If F is total, then $\text{rng}\, F \upharpoonleft F$ is total. The theorem is a consequence of (85).

(87) If F is onto, then $F \restriction \text{dom}\, F$ is onto. The theorem is a consequence of (85).

Let us consider graphs G_1, G_2. Now we state the propositions:

(88) Every partial graph mapping from G_1 to G_2 is a partial graph mapping from G_1 to $\text{rng}\, F$. The theorem is a consequence of (85).

(89) Every partial graph mapping from G_1 to G_2 is a partial graph mapping from $\mathrm{dom}\, F$ to G_2. The theorem is a consequence of (85).

(90) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is total. Then $(F_{\mathbb{E}})^{\circ}(G_1.\mathrm{edgesBetween}(X, Y)) \subseteq G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)$.
PROOF: Set $f = F_{\mathbb{E}} {\restriction} G_1.\mathrm{edgesBetween}(X, Y)$. For every object y such that $y \in \mathrm{rng}\, f$ holds $y \in G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)$. $\square$

(91) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a set V. Then $(F_{\mathbb{E}})^{\circ}(G_1.\mathrm{edgesBetween}(V)) \subseteq G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}V)$.

(92) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and subsets X, Y of the vertices of G_1. Suppose F is weak subgraph embedding and onto.
Then $(F_{\mathbb{E}})^{\circ}(G_1.\mathrm{edgesBetween}(X, Y)) = G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}X, (F_{\mathbb{V}})^{\circ}Y)$. The theorem is a consequence of (90).

(93) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a set V. Suppose F is continuous. Then $(F_{\mathbb{E}})^{\circ}(G_1.\mathrm{edgesBetween}(V)) = G_2.\mathrm{edgesBetween}((F_{\mathbb{V}})^{\circ}V)$. The theorem is a consequence of (91).

Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an F-valued walk W_2 of G_2. Now we state the propositions:

(94) $(F^{-1}(W_2)).\mathrm{vertices}() = (F_{\mathbb{V}})^{-1}(W_2.\mathrm{vertices}())$.

(95) $(F^{-1}(W_2)).\mathrm{edges}() = (F_{\mathbb{E}})^{-1}(W_2.\mathrm{edges}())$.

(96) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, an F-valued walk W_2 of G_2, and objects v, w. Suppose W_2 is a walk from v to w. Then $F^{-1}(W_2)$ is a walk from $(F^{-1}{}_{\mathbb{V}})(v)$ to $(F^{-1}{}_{\mathbb{V}})(w)$.

(97) Let us consider graphs G_1, G_2, a one-to-one partial graph mapping F from G_1 to G_2, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_2 = (F_{\mathbb{V}})(v_1)$ and F is isomorphism. Then $(F_{\mathbb{V}})^{-1}(G_2.\mathrm{reachableFrom}(v_2)) = G_1.\mathrm{reachableFrom}(v_1)$. The theorem is a consequence of (81).

(98) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a subgraph H of G_2. Then $(F_{\mathbb{E}})^{-1}(\text{the edges of } H) \subseteq G_1.\mathrm{edgesBetween}$ $((F_{\mathbb{V}})^{-1}(\text{the vertices of } H))$.

(99) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, a subgraph H_2 of $\mathrm{rng}\, F$, and a subgraph H_1 of G_1 induced by $(F_{\mathbb{V}})^{-1}(\text{the vertices of } H_2)$ and $(F_{\mathbb{E}})^{-1}(\text{the edges of } H_2)$. Then $\mathrm{rng}(F {\restriction} H_1) \approx H_2$. The theorem is a consequence of (98).

(100) Let us consider graphs G_1, G_2, a non empty partial graph mapping F

from G_1 to G_2, a non empty subset V_2 of the vertices of rng F, and a subgraph H of rng F induced by V_2. Suppose G_1.edgesBetween($(F_{\mathbb{V}})^{-1}$(the vertices of H)) $\subseteq$ dom($F_{\mathbb{E}}$). Then $(F_{\mathbb{E}})^{-1}$(the edges of H) $= G_1$.edges Between($(F_{\mathbb{V}})^{-1}$(the vertices of H)). The theorem is a consequence of (98).

(101) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, a non empty subset V_2 of the vertices of rng F, a subgraph H_2 of rng F induced by V_2, and a subgraph H_1 of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2). Suppose G_1.edgesBetween($(F_{\mathbb{V}})^{-1}$(the vertices of H_2)) $\subseteq$ dom($F_{\mathbb{E}}$). Then rng($F{\restriction}H_1$) $\approx H_2$. The theorem is a consequence of (100).

(102) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, a non empty subset V of the vertices of dom F, and a subgraph H of G_1 induced by V. Suppose F is continuous. Then rng($F{\restriction}H$) is a subgraph of G_2 induced by $(F_{\mathbb{V}})^{\circ}V$. The theorem is a consequence of (93).

(103) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, a subgraph H_2 of rng F, and a subgraph H_1 of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2) and $(F_{\mathbb{E}})^{-1}$(the edges of H_2). Then every walk of H_1 is an F-defined walk of G_1.

PROOF: the vertices of $H_1 = (F_{\mathbb{V}})^{-1}$(the vertices of H_2) and the edges of $H_1 = (F_{\mathbb{E}})^{-1}$(the edges of H_2). □

(104) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, a subgraph H_2 of rng F, a subgraph H_1 of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2) and $(F_{\mathbb{E}})^{-1}$(the edges of H_2), and an F-defined walk W_1 of G_1. If W_1 is a walk of H_1, then $F^{\circ}W_1$ is a walk of H_2.

PROOF: the vertices of $H_1 = (F_{\mathbb{V}})^{-1}$(the vertices of H_2) and the edges of $H_1 = (F_{\mathbb{E}})^{-1}$(the edges of H_2). $(F^{\circ}W_1)$.vertices() $\subseteq$ the vertices of H_2. $(F^{\circ}W_1)$.edges() $\subseteq$ the edges of H_2. □

(105) Let us consider graphs G_1, G_2, a non empty partial graph mapping F from G_1 to G_2, and a subgraph H of rng F. Then every walk of H is an F-valued walk of G_2.

(106) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, a subgraph H_2 of rng F, a subgraph H_1 of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2) and $(F_{\mathbb{E}})^{-1}$(the edges of H_2), and an F-valued walk W_2 of G_2. If W_2 is a walk of H_2, then $F^{-1}(W_2)$ is a walk of H_1.

PROOF: the vertices of $H_1 = (F_{\mathbb{V}})^{-1}$(the vertices of H_2) and the edges of $H_1 = (F_{\mathbb{E}})^{-1}$(the edges of H_2). $(F^{-1}(W_2))$.vertices() $\subseteq$ the vertices of H_1. $(F^{-1}(W_2))$.edges() $\subseteq$ the edges of H_1. □

(107) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and an acyclic subgraph H_2 of rng F. Then every subgraph of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2) and $(F_{\mathbb{E}})^{-1}$(the edges of H_2) is acyclic. The theorem is a consequence of (103) and (104).

(108) Let us consider graphs G_1, G_2, a non empty, one-to-one partial graph mapping F from G_1 to G_2, and a connected subgraph H_2 of rng F. Then every subgraph of G_1 induced by $(F_{\mathbb{V}})^{-1}$(the vertices of H_2) and $(F_{\mathbb{E}})^{-1}$(the edges of H_2) is connected. The theorem is a consequence of (98), (105), (106), and (96).

Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, a subgraph H of G_1, and a partial graph mapping F' from H to $\mathrm{rng}(F{\restriction}H)$. Now we state the propositions:

(109) Suppose $F' = F{\restriction}H$. Then

- (i) if F' is not empty, then F' is onto, and
- (ii) if F is total, then F' is total, and
- (iii) if F is one-to-one, then F' is one-to-one, and
- (iv) if F is directed, then F' is directed, and
- (v) if F is semi-continuous, then F' is semi-continuous, and
- (vi) if F is continuous and $F_{\mathbb{E}}$ is one-to-one, then F' is continuous, and
- (vii) if F is semi-directed-continuous, then F' is semi-directed-continuous, and
- (viii) if F is directed-continuous and $F_{\mathbb{E}}$ is one-to-one, then F' is directed-continuous.

The theorem is a consequence of (85) and (86).

(110) Suppose $F' = F{\restriction}H$. Then

- (i) if F is weak subgraph embedding, then F' is weak subgraph embedding, and
- (ii) if F is strong subgraph embedding, then F' is isomorphism, and
- (iii) if F is directed and strong subgraph embedding, then F' is directed-isomorphism.

The theorem is a consequence of (109).

11. Into GLIB_013

Now we state the propositions:

(111) Let us consider a vertex-finite, directed-simple graph G_1, a directed graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. Suppose $v_1 = v_2$. Then

(i) v_2.inDegree() $= G_1$.order() $-$ (v_1.inDegree() $+ 1$), and

(ii) v_2.outDegree() $= G_1$.order() $-$ (v_1.outDegree() $+ 1$), and

(iii) v_2.degree() $= 2 \cdot (G_1$.order()$)$ $-$ (v_1.degree() $+ 2$).

(112) Let us consider a vertex-finite, simple graph G_1, a graph complement G_2 of G_1, a vertex v_1 of G_1, and a vertex v_2 of G_2. If $v_1 = v_2$, then v_2.degree() $= G_1$.order() $-$ (v_1.degree() $+ 1$).

(113) Let us consider a vertex-finite, directed-simple graph G, and a vertex v of G. Then

(i) v.inDegree() $<$ G.order(), and

(ii) v.outDegree() $<$ G.order().

(114) Let us consider a vertex-finite, simple graph G, and a vertex v of G. Then v.degree() $<$ G.order().

One can check that every graph which is 1-edge is also non-multi.

12. Into GLIB_014

Let S be a $\cup$-tolerating, graph-membered set. Observe that every subset of S is $\cup$-tolerating.

Now we state the proposition:

(115) Let us consider graph-membered sets S_1, S_2. Suppose $S_1 \subseteq S_2$. Then

(i) the vertices of $S_1 \subseteq$ the vertices of S_2, and

(ii) the edges of $S_1 \subseteq$ the edges of S_2, and

(iii) the source of $S_1 \subseteq$ the source of S_2, and

(iv) the target of $S_1 \subseteq$ the target of S_2.

Let us consider a graph union set S, a graph union G of S, and objects e, v, w. Now we state the propositions:

(116) If e joins v to w in G, then there exists an element H of S such that e joins v to w in H.

(117) If e joins v and w in G, then there exists an element H of S such that e joins v and w in H. The theorem is a consequence of (116).

Let us consider graph union sets S_1, S_2, a graph union G_1 of S_1, and a graph union G_2 of S_2. Now we state the propositions:

(118) If for every element H_2 of S_2, there exists an element H_1 of S_1 such that H_2 is a subgraph of H_1, then G_2 is a subgraph of G_1. The theorem is a consequence of (116).

(119) If $S_2 \subseteq S_1$, then G_2 is a subgraph of G_1. The theorem is a consequence of (118).

Let us consider graphs G_1, G_2 and a graph union G of G_1 and G_2. Now we state the propositions:

(120) If G_1 tolerates G_2 and the vertices of G_1 misses the vertices of G_2, then $G.\text{order}() = G_1.\text{order}() + G_2.\text{order}()$.

(121) If G_1 tolerates G_2 and the edges of G_1 misses the edges of G_2, then $G.\text{size}() = G_1.\text{size}() + G_2.\text{size}()$.

(122) Let us consider connected graphs G_1, G_2, and a graph union G of G_1 and G_2. If the vertices of G_1 meets the vertices of G_2, then G is connected.

(123) Let us consider graphs G_1, G_2, a graph union G of G_1 and G_2, and a walk W of G. Suppose G_1 tolerates G_2 and the vertices of G_1 misses the vertices of G_2. Then W is a walk of G_1 or a walk of G_2.

(124) Let us consider graphs G_1, G_2, a graph union G of G_1 and G_2, a vertex v_1 of G_1, and a vertex v of G. Suppose the vertices of G_1 misses the vertices of G_2. If $v = v_1$, then $G.\text{reachableFrom}(v) = G_1.\text{reachableFrom}(v_1)$. The theorem is a consequence of (123).

(125) Let us consider graphs G_1, G_2, a graph union G of G_1 and G_2, a vertex v_2 of G_2, and a vertex v of G. Suppose G_1 tolerates G_2 and the vertices of G_1 misses the vertices of G_2. If $v = v_2$, then $G.\text{reachableFrom}(v) = G_2.\text{reachableFrom}(v_2)$. The theorem is a consequence of (123).

(126) Let us consider graphs G_1, G_2, and a graph union G of G_1 and G_2. Suppose G_1 tolerates G_2 and the vertices of G_1 misses the vertices of G_2. Then

(i) $G.\text{componentSet}() = G_1.\text{componentSet}() \cup G_2.\text{componentSet}()$, and

(ii) $G.\text{numComponents}() = G_1.\text{numComponents}() + G_2.\text{numComponents}()$.

The theorem is a consequence of (124) and (125).

13. Into GLUNIR00

Let us consider a non empty set V and a binary relation E on V. Now we state the propositions:

(127) $\text{createGraph}(V, E).\text{loops}() = E \cap \text{id}_V$.

(128) $\text{createGraph}(V, E \setminus (\text{id}_V))$ is a subgraph of $\text{createGraph}(V, E)$ with loops removed. The theorem is a consequence of (127).

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