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Inverse Function Theorem. Part I[1]

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Summary. In this article we formalize in Mizar [1], [2] the inverse function theorem for the class of C^1 functions between Banach spaces. In the first section, we prove several theorems about open sets in real norm space, which are needed in the proof of the inverse function theorem. In the next section, we define a function to exchange the order of a product of two normed spaces, namely $\mathbb{E}\curvearrowleft\approx(x, y) \in X \times Y \mapsto (y, x) \in Y \times X$, and formalized its bijective isometric property and several differentiation properties. This map is necessary to change the order of the arguments of a function when deriving the inverse function theorem from the implicit function theorem proved in [6].

In the third section, using the implicit function theorem, we prove a theorem that is a necessary component of the proof of the inverse function theorem. In the last section, we finally formalized an inverse function theorem for class of C^1 functions between Banach spaces. We referred to [9], [10], and [3] in the formalization.

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1. Preliminaries

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

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(1) Let us consider real normed spaces X, Y, a partial function f from X to Y, a subset A of X, and a subset B of Y. Suppose $\operatorname{dom} f = A$ and f is continuous on A and A is open and B is open. Then $f^{-1}(B)$ is open.
PROOF: For every point a of X such that $a \in f^{-1}(B)$ there exists a real number s such that $s > 0$ and $\operatorname{Ball}(a, s) \subseteq f^{-1}(B)$. □

(2) Let us consider real normed spaces X, Y, a point x of X, a point y of Y, a point z of $X \times Y$, and real numbers r_1, r_2. Suppose $0 < r_1$ and $0 < r_2$ and $z = \langle x, y \rangle$. Then there exists a real number s such that

  (i) $s = \min(r_1, r_2)$, and

  (ii) $s > 0$, and

  (iii) $\operatorname{Ball}(z, s) \subseteq \operatorname{Ball}(x, r_1) \times \operatorname{Ball}(y, r_2)$.

(3) Let us consider real normed spaces X, Y, and a subset V of $X \times Y$. Then V is open if and only if for every point x of X and for every point y of Y such that $\langle x, y \rangle \in V$ there exist real numbers r_1, r_2 such that $0 < r_1$ and $0 < r_2$ and $\operatorname{Ball}(x, r_1) \times \operatorname{Ball}(y, r_2) \subseteq V$.
PROOF: For every point z of $X \times Y$ such that $z \in V$ there exists a real number s such that $s > 0$ and $\operatorname{Ball}(z, s) \subseteq V$. □

(4) Let us consider real normed spaces X, Y, a subset V of $X \times Y$, and a subset D of X. Suppose D is open and $V = D \times$ (the carrier of Y). Then V is open.
PROOF: For every point x of X and for every point y of Y such that $\langle x, y \rangle \in V$ there exist real numbers r_1, r_2 such that $0 < r_1$ and $0 < r_2$ and $\operatorname{Ball}(x, r_1) \times \operatorname{Ball}(y, r_2) \subseteq V$. □

(5) Let us consider real normed spaces X, Y, a subset V of $X \times Y$, and a subset D of Y. Suppose D is open and $V =$ (the carrier of $X) \times D$. Then V is open.
PROOF: For every point x of X and for every point y of Y such that $\langle x, y \rangle \in V$ there exist real numbers r_1, r_2 such that $0 < r_1$ and $0 < r_2$ and $\operatorname{Ball}(x, r_1) \times \operatorname{Ball}(y, r_2) \subseteq V$. □

2. A Map Reversing the Order of Product of Two Norm Spaces

Now we state the proposition:

(6) Let us consider real numbers x, y, and elements u, v of $\mathcal{R}^2$. Suppose $u = \langle x, y \rangle$ and $v = \langle y, x \rangle$. Then $|u| = |v|$.

Let X, Y be real normed spaces. The functor $\operatorname{Exch}(X, Y)$ yielding a linear operator from $X \times Y$ into $Y \times X$ is defined by

(Def. 1) *it* is one-to-one, onto, and isometric and for every point x of X and for every point y of Y, $it(x, y) = \langle y, x \rangle$.

Now we state the propositions:

(7) Let us consider real normed spaces X, Y, a subset Z of $X \times Y$, and objects x, y. Then $\langle x, y \rangle \in Z$ if and only if $\langle y, x \rangle \in (\mathrm{Exch}(Y, X))^{-1}(Z)$.

(8) Let us consider real normed spaces X, Y, a non empty set Z, a partial function f from $X \times Y$ to Z, and a function I from $Y \times X$ into $X \times Y$. Suppose for every point y of Y for every point x of X, $I(y, x) = \langle x, y \rangle$. Then

(i) $\mathrm{dom}(f \cdot I) = I^{-1}(\mathrm{dom}\, f)$, and

(ii) for every point x of X and for every point y of Y, $f \cdot I(y, x) = f(x, y)$.

PROOF: For every object w, $w \in \mathrm{dom}(f \cdot I)$ iff $w \in I^{-1}(\mathrm{dom}\, f)$. □

(9) Let us consider real normed spaces X, Y, Z, a partial function f from Y to Z, a linear operator I from X into Y, and a subset V of Y. Suppose f is differentiable on V and I is one-to-one, onto, and isometric. Let us consider a point y of Y. Suppose $y \in V$. Then $(f'_{\restriction V})(y) = (f \cdot I'_{\restriction I^{-1}(V)})_{/(I^{-1})(y)} \cdot (I^{-1})$.
PROOF: Consider J being a linear operator from Y into X such that $J = I^{-1}$ and J is one-to-one, onto, and isometric. Set $g = f \cdot I$. Set $U = I^{-1}(V)$. For every point y of Y such that $y \in \mathrm{dom}(f'_{\restriction V})$ holds $(f'_{\restriction V})(y) = (g'_{\restriction U})_{/J(y)} \cdot (I^{-1})$ by [4, (31)]. □

(10) Let us consider real normed spaces X, Y, Z, a subset V of Y, a partial function g from Y to Z, and a linear operator I from X into Y. Suppose I is one-to-one, onto, and isometric and g is differentiable on V. Then $g'_{\restriction V}$ is continuous on V if and only if $g \cdot I'_{\restriction I^{-1}(V)}$ is continuous on $I^{-1}(V)$.
PROOF: Consider J being a linear operator from Y into X such that $J = I^{-1}$ and J is one-to-one, onto, and isometric. Set $f = g \cdot I$. Set $U = I^{-1}(V)$. Set $F = f'_{\restriction U}$. Set $G = g'_{\restriction V}$. If G is continuous on V, then F is continuous on U. If F is continuous on U, then G is continuous on V. □

(11) Let us consider real normed spaces X, Y, Z, a partial function f from $X \times Y$ to Z, a subset U of $X \times Y$, and a function I from $Y \times X$ into $X \times Y$. Suppose for every point y of Y for every point x of X, $I(y, x) = \langle x, y \rangle$. Let us consider a point a of X, a point b of Y, a point u of $X \times Y$, and a point v of $Y \times X$. Suppose $u \in U$ and $u = \langle a, b \rangle$ and $v = \langle b, a \rangle$. Then

(i) $f \cdot (\mathrm{reproj1}(u)) = f \cdot I \cdot (\mathrm{reproj2}(v))$, and

(ii) $f \cdot (\mathrm{reproj2}(u)) = f \cdot I \cdot (\mathrm{reproj1}(v))$.

PROOF: For every object x, $x \in \mathrm{dom}(f \cdot (\mathrm{reproj1}(u)))$ iff $x \in \mathrm{dom}(f \cdot I \cdot (\mathrm{reproj2}(v)))$. For every object y, $y \in \mathrm{dom}(f \cdot (\mathrm{reproj2}(u)))$ iff $y \in \mathrm{dom}(f \cdot$

$I \cdot (\text{reproj1}(v))$. For every object x such that $x \in \text{dom}(f \cdot (\text{reproj1}(u)))$ holds $(f \cdot (\text{reproj1}(u)))(x) = (f \cdot I \cdot (\text{reproj2}(v)))(x)$. For every object y such that $y \in \text{dom}(f \cdot (\text{reproj2}(u)))$ holds $(f \cdot (\text{reproj2}(u)))(y) = (f \cdot I \cdot (\text{reproj1}(v)))(y)$. □

Let us consider real normed spaces X, Y, Z, a partial function f from $X \times Y$ to Z, a subset U of $X \times Y$, a linear operator I from $Y \times X$ into $X \times Y$, a point a of X, a point b of Y, a point u of $X \times Y$, and a point v of $Y \times X$. Now we state the propositions:

(12) Suppose $U = \text{dom}\, f$ and f is differentiable on U and I is one-to-one, onto, and isometric and for every point y of Y and for every point x of X, $I(y, x) = \langle x, y \rangle$. Then suppose $u \in U$ and $u = \langle a, b \rangle$ and $v = \langle b, a \rangle$. Then

- (i) f is partially differentiable in u w.r.t. 1 iff $f \cdot I$ is partially differentiable in v w.r.t. 2, and
- (ii) f is partially differentiable in u w.r.t. 2 iff $f \cdot I$ is partially differentiable in v w.r.t. 1.

(13) Suppose $U = \text{dom}\, f$ and f is differentiable on U and I is one-to-one, onto, and isometric and for every point y of Y and for every point x of X, $I(y, x) = \langle x, y \rangle$. Then suppose $u \in U$ and $u = \langle a, b \rangle$ and $v = \langle b, a \rangle$. Then

- (i) $\text{partdiff}(f, u)\,\text{w.r.t.}\,1 = \text{partdiff}(f \cdot I, v)\,\text{w.r.t.}\,2$, and
- (ii) $\text{partdiff}(f, u)\,\text{w.r.t.}\,2 = \text{partdiff}(f \cdot I, v)\,\text{w.r.t.}\,1$.

3. Properties of the Differentiation of the Inverse Mapping

Now we state the propositions:

(14) Let us consider a real normed space F, non trivial real Banach spaces G, E, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point a of E, a point b of F, a point c of G, and a point z of $E \times F$. Suppose Z is open and $\text{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $\langle a, b \rangle \in Z$ and $f(a, b) = c$ and $z = \langle a, b \rangle$ and $\text{partdiff}(f, z)\,\text{w.r.t.}\,1$ is invertible. Then there exist real numbers r_1, r_2 such that

- (i) $0 < r_1$, and
- (ii) $0 < r_2$, and
- (iii) $\overline{\text{Ball}}(a, r_1) \times \text{Ball}(b, r_2) \subseteq Z$, and
- (iv) for every point y of F such that $y \in \text{Ball}(b, r_2)$ there exists a point x of E such that $x \in \text{Ball}(a, r_1)$ and $f(x, y) = c$, and

(v) for every point y of F such that $y \in \text{Ball}(b, r_2)$ for every points x_1, x_2 of E such that x_1, $x_2 \in \text{Ball}(a, r_1)$ and $f(x_1, y) = c$ and $f(x_2, y) = c$ holds $x_1 = x_2$, and

(vi) there exists a partial function g from F to E such that $\text{dom}\, g = \text{Ball}(b, r_2)$ and $\text{rng}\, g \subseteq \text{Ball}(a, r_1)$ and g is continuous on $\text{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f(g(y), y) = c$ and g is differentiable on $\text{Ball}(b, r_2)$ and $g'_{\restriction \text{Ball}(b, r_2)}$ is continuous on $\text{Ball}(b, r_2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \text{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds $g'(y) = -(\text{Inv partdiff}(f, z) \text{ w.r.t. } 1) \cdot (\text{partdiff}(f, z) \text{ w.r.t. } 2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \text{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds $\text{partdiff}(f, z)$ w.r.t. 1 is invertible, and

(vii) for every partial functions g_1, g_2 from F to E such that $\text{dom}\, g_1 = \text{Ball}(b, r_2)$ and $\text{rng}\, g_1 \subseteq \text{Ball}(a, r_1)$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f(g_1(y), y) = c$ and $\text{dom}\, g_2 = \text{Ball}(b, r_2)$ and $\text{rng}\, g_2 \subseteq \text{Ball}(a, r_1)$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f(g_2(y), y) = c$ holds $g_1 = g_2$.

PROOF: Set $I = \text{Exch}(F, E)$. Consider J being a linear operator from $E \times F$ into $F \times E$ such that $J = I^{-1}$ and J is one-to-one, onto, and isometric. Set $Z_1 = J^{\circ} Z$. Set $f_1 = f \cdot I$. $\text{dom}\, f_1 = I^{-1}(\text{dom}\, f)$. Reconsider $z_1 = \langle b, a \rangle$ as a point of $F \times E$. ${f_1}'_{\restriction Z_1}$ is continuous on Z_1. $f_1(b, a) = c$. $\text{partdiff}(f, z)$ w.r.t. $1 = \text{partdiff}(f_1, z_1)$ w.r.t. 2. Consider r_2, r_1 being real numbers such that $0 < r_2$ and $0 < r_1$ and $\text{Ball}(b, r_2) \times \overline{\text{Ball}}(a, r_1) \subseteq Z_1$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ there exists a point x of E such that $x \in \text{Ball}(a, r_1)$ and $f_1(y, x) = c$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ for every points x_1, x_2 of E such that x_1, $x_2 \in \text{Ball}(a, r_1)$ and $f_1(y, x_1) = c$ and $f_1(y, x_2) = c$ holds $x_1 = x_2$ and there exists a partial function g from F to E such that $\text{dom}\, g = \text{Ball}(b, r_2)$ and $\text{rng}\, g \subseteq \text{Ball}(a, r_1)$ and g is continuous on $\text{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f_1(y, g(y)) = c$.

g is differentiable on $\text{Ball}(b, r_2)$ and $g'_{\restriction \text{Ball}(b, r_2)}$ is continuous on $\text{Ball}(b, r_2)$ and for every point y of F and for every point z of $F \times E$ such that $y \in \text{Ball}(b, r_2)$ and $z = \langle y, g(y) \rangle$ holds $g'(y) = -(\text{Inv partdiff}(f_1, z) \text{ w.r.t. } 2) \cdot (\text{partdiff}(f_1, z) \text{ w.r.t. } 1)$ and for every point y of F and for every point z of $F \times E$ such that $y \in \text{Ball}(b, r_2)$ and $z = \langle y, g(y) \rangle$ holds $\text{partdiff}(f_1, z)$ w.r.t. 2 is invertible.

For every partial functions g_1, g_2 from F to E such that $\text{dom}\, g_1 = \text{Ball}(b, r_2)$ and $\text{rng}\, g_1 \subseteq \text{Ball}(a, r_1)$ and for every point y of F such that

$y \in \mathrm{Ball}(b, r_2)$ holds $f_1(y, g_1(y)) = c$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_1(y, g_2(y)) = c$ holds $g_1 = g_2$. For every object s such that $s \in \overline{\mathrm{Ball}}(a, r_1) \times \mathrm{Ball}(b, r_2)$ holds $s \in Z$. For every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ there exists a point x of E such that $x \in \mathrm{Ball}(a, r_1)$ and $f(x, y) = c$.

For every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ for every points x_1, x_2 of E such that x_1, $x_2 \in \mathrm{Ball}(a, r_1)$ and $f(x_1, y) = c$ and $f(x_2, y) = c$ holds $x_1 = x_2$. There exists a partial function g from F to E such that $\mathrm{dom}\, g = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(a, r_1)$ and g is continuous on $\mathrm{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f(g(y), y) = c$.

g is differentiable on $\mathrm{Ball}(b, r_2)$ and $g'_{\restriction \mathrm{Ball}(b, r_2)}$ is continuous on $\mathrm{Ball}(b, r_2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \mathrm{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds $g'(y) = -(\mathrm{Inv}\, \mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1) \cdot (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \mathrm{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds $\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1$ is invertible.

For every partial functions g_1, g_2 from F to E such that $\mathrm{dom}\, g_1 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f(g_1(y), y) = c$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f(g_2(y), y) = c$ holds $g_1 = g_2$. □

(15) Let us consider non trivial real Banach spaces E, F, a subset D of E, a partial function f from E to F, a partial function f_1 from $E \times F$ to F, and a subset Z of $E \times F$. Suppose D is open and $\mathrm{dom}\, f = D$ and $D \neq \emptyset$ and f is differentiable on D and $f'_{\restriction D}$ is continuous on D and $Z = D \times$ (the carrier of F) and $\mathrm{dom}\, f_1 = Z$ and for every point s of E and for every point t of F such that $s \in D$ holds $f_1(s, t) = f_{/s} - t$. Then

(i) f_1 is differentiable on Z, and

(ii) ${f_1}'_{\restriction Z}$ is continuous on Z, and

(iii) for every point x of E and for every point y of F and for every point z of $E \times F$ such that $x \in D$ and $z = \langle x, y \rangle$ there exists a point I of the real norm space of bounded linear operators from F into F such that $I = \mathrm{id}_\alpha$ and $\mathrm{partdiff}(f_1, z)\ \mathrm{w.r.t.}\ 1 = f'(x)$ and $\mathrm{partdiff}(f_1, z)\ \mathrm{w.r.t.}\ 2 = -I$,

where α is the carrier of F.

Proof: Z is open. For every point z of $E \times F$ such that $z \in Z$ holds f_1 is partially differentiable in z w.r.t. 1 and $\mathrm{partdiff}(f_1, z)\ \mathrm{w.r.t.}\ 1 = f'((z)_{\mathbf{1}})$. For every point x_0 of $E \times F$ and for every real number r such that $x_0 \in Z$

and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $E \times F$ such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|(f_1 \restriction^1 Z)_{/x_1} - (f_1 \restriction^1 Z)_{/x_0}\| < r$ by [8, (15)]. Reconsider $J = \mathrm{FuncUnit}(F)$ as a point of the real norm space of bounded linear operators from F into F.

For every point z of $E \times F$ such that $z \in Z$ holds f_1 is partially differentiable in z w.r.t. 2 and $\mathrm{partdiff}(f_1, z)$ w.r.t. $2 = -J$. For every point x_0 of $E \times F$ and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $E \times F$ such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|(f_1 \restriction^2 Z)_{/x_1} - (f_1 \restriction^2 Z)_{/x_0}\| < r$. For every point x of E and for every point y of F and for every point z of $E \times F$ such that $x \in D$ and $z = \langle x, y \rangle$ there exists a point I of the real norm space of bounded linear operators from F into F such that $I = \mathrm{id}_\alpha$ and $\mathrm{partdiff}(f_1, z)$ w.r.t. $1 = f'(x)$ and $\mathrm{partdiff}(f_1, z)$ w.r.t. $2 = -I$, where α is the carrier of F. □

(16) Let us consider non trivial real Banach spaces E, F, a subset Z of E, a partial function f from E to F, a point a of E, and a point b of F. Suppose Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\overline{\mathrm{Ball}}(a, r_1) \subseteq Z$, and

(iv) for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ there exists a point x of E such that $x \in \mathrm{Ball}(a, r_1)$ and $f_{/x} = y$, and

(v) for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ for every points x_1, x_2 of E such that $x_1, x_2 \in \mathrm{Ball}(a, r_1)$ and $f_{/x_1} = y$ and $f_{/x_2} = y$ holds $x_1 = x_2$, and

(vi) there exists a partial function g from F to E such that $\mathrm{dom}\, g = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(a, r_1)$ and g is continuous on $\mathrm{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_{/g_{/y}} = y$ and g is differentiable on $\mathrm{Ball}(b, r_2)$ and $g'_{\restriction \mathrm{Ball}(b, r_2)}$ is continuous on $\mathrm{Ball}(b, r_2)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $g'(y) = \mathrm{Inv}\, f'(g_{/y})$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f'(g_{/y})$ is invertible, and

(vii) for every partial functions g_1, g_2 from F to E such that $\mathrm{dom}\, g_1 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_{/g_1(y)} = y$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_{/g_2(y)} = y$ holds $g_1 = g_2$.

PROOF: Reconsider $Z = D \times$ (the carrier of F) as a subset of $E \times F$. Z is open. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point x of E and there exists a point y of F such that $\$_1 = \langle x, y \rangle$ and $\$_2 = f_{/x} - y$. For every object z such that $z \in Z$ there exists an object y such that $y \in$ the carrier of F and $\mathcal{P}[z, y]$.

Consider f_1 being a function from Z into the carrier of F such that for every object x such that $x \in Z$ holds $\mathcal{P}[x, f_1(x)]$. For every point s of E and for every point t of F such that $s \in D$ holds $f_1(s, t) = f_{/s} - t$. Reconsider $z = \langle a, b \rangle$ as a point of $E \times F$. f_1 is differentiable on Z. ${f_1}'_{\restriction Z}$ is continuous on Z. There exists a point J of the real norm space of bounded linear operators from F into F such that $J = \mathrm{id}_\alpha$ and partdiff(f_1, z) w.r.t. $1 = f'(a)$ and partdiff(f_1, z) w.r.t. $2 = -J$, where α is the carrier of F.

Consider r_1, r_2 being real numbers such that $0 < r_1$ and $0 < r_2$ and $\overline{\mathrm{Ball}}(a, r_1) \times \mathrm{Ball}(b, r_2) \subseteq Z$ and for every point x of F such that $x \in \mathrm{Ball}(b, r_2)$ there exists a point y of E such that $y \in \mathrm{Ball}(a, r_1)$ and $f_1(y, x) = 0_F$ and for every point x of F such that $x \in \mathrm{Ball}(b, r_2)$ for every points y_1, y_2 of E such that y_1, $y_2 \in \mathrm{Ball}(a, r_1)$ and $f_1(y_1, x) = 0_F$ and $f_1(y_2, x) = 0_F$ holds $y_1 = y_2$ and there exists a partial function g from F to E such that $\mathrm{dom}\, g = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(a, r_1)$ and g is continuous on $\mathrm{Ball}(b, r_2)$ and $g(b) = a$ and for every point x of F such that $x \in \mathrm{Ball}(b, r_2)$ holds $f_1(g(x), x) = 0_F$ and g is differentiable on $\mathrm{Ball}(b, r_2)$.

$g'_{\restriction \mathrm{Ball}(b, r_2)}$ is continuous on $\mathrm{Ball}(b, r_2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \mathrm{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds $g'(y) = -(\mathrm{Inv}\, \text{partdiff}(f_1, z) \text{ w.r.t. } 1) \cdot (\text{partdiff}(f_1, z) \text{ w.r.t. } 2)$ and for every point y of F and for every point z of $E \times F$ such that $y \in \mathrm{Ball}(b, r_2)$ and $z = \langle g(y), y \rangle$ holds partdiff(f_1, z) w.r.t. 1 is invertible and for every partial functions g_1, g_2 from F to E such that $\mathrm{dom}\, g_1 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_1(g_1(y), y) = 0_F$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(a, r_1)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_1(g_2(y), y) = 0_F$ holds $g_1 = g_2$. For every object s such that $s \in \overline{\mathrm{Ball}}(a, r_1)$ holds $s \in D$. For every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ there exists a point x of E such that $x \in \mathrm{Ball}(a, r_1)$ and $f_{/x} = y$. For every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ for every points x_1, x_2 of E such that x_1, $x_2 \in \mathrm{Ball}(a, r_1)$ and $f_{/x_1} = y$ and $f_{/x_2} = y$ holds $x_1 = x_2$.

There exists a partial function g from F to E such that $\mathrm{dom}\, g = \mathrm{Ball}(b, r_2)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(a, r_1)$ and g is continuous on $\mathrm{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds $f_{/g_{/y}} = y$ and g is differentiable on $\mathrm{Ball}(b, r_2)$ and $g'_{\restriction \mathrm{Ball}(b, r_2)}$ is continuous on $\mathrm{Ball}(b, r_2)$ and for every point y of F such that $y \in \mathrm{Ball}(b, r_2)$ holds

$g'(y) = \operatorname{Inv} f'(g_{/y})$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f'(g_{/y})$ is invertible by (15), [5, (26),(27)]. For every partial functions g_1, g_2 from F to E such that $\operatorname{dom} g_1 = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(a, r_1)$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f_{/g_1(y)} = y$ and $\operatorname{dom} g_2 = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(a, r_1)$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f_{/g_2(y)} = y$ holds $g_1 = g_2$. □

4. Inverse Function Theorem for Class of C^1 Functions

Now we state the propositions:

(17) Let us consider non trivial real Banach spaces E, F, a subset Z of E, a partial function f from E to F, a point a of E, and a point b of F. Suppose Z is open and $\operatorname{dom} f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible.

Then there exists a subset A of E and there exists a subset B of F and there exists a partial function g from F to E such that A is open and B is open and $A \subseteq \operatorname{dom} f$ and $a \in A$ and $b \in B$ and $f^\circ A = B$ and $\operatorname{dom} g = B$ and $\operatorname{rng} g = A$ and $\operatorname{dom}(f{\restriction}A) = A$ and $\operatorname{rng}(f{\restriction}A) = B$ and $f{\restriction}A$ is one-to-one and g is one-to-one and $g = (f{\restriction}A)^{-1}$ and $f{\restriction}A = g^{-1}$ and $g(b) = a$ and g is continuous on B and differentiable on B and $g'_{\restriction B}$ is continuous on B and for every point y of F such that $y \in B$ holds $f'(g_{/y})$ is invertible and for every point y of F such that $y \in B$ holds $g'(y) = \operatorname{Inv} f'(g_{/y})$.

Proof: Consider r_1, r_2 being real numbers such that $0 < r_1$ and $0 < r_2$ and $\overline{\operatorname{Ball}}(a, r_1) \subseteq Z$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ there exists a point x of E such that $x \in \operatorname{Ball}(a, r_1)$ and $f_{/x} = y$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ for every points x_1, x_2 of E such that $x_1, x_2 \in \operatorname{Ball}(a, r_1)$ and $f_{/x_1} = y$ and $f_{/x_2} = y$ holds $x_1 = x_2$ and there exists a partial function g from F to E such that $\operatorname{dom} g = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} g \subseteq \operatorname{Ball}(a, r_1)$ and g is continuous on $\operatorname{Ball}(b, r_2)$ and $g(b) = a$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f_{/g_{/y}} = y$.

g is differentiable on $\operatorname{Ball}(b, r_2)$ and $g'_{\restriction \operatorname{Ball}(b, r_2)}$ is continuous on $\operatorname{Ball}(b, r_2)$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $g'(y) = \operatorname{Inv} f'(g_{/y})$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f'(g_{/y})$ is invertible and for every partial functions g_1, g_2 from F to E such that $\operatorname{dom} g_1 = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(a, r_1)$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f_{/g_1(y)} = y$ and $\operatorname{dom} g_2 = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(a, r_1)$ and for every point y of F such that $y \in \operatorname{Ball}(b, r_2)$ holds $f_{/g_2(y)} = y$ holds $g_1 = g_2$.

Consider I_1 being a partial function from F to E such that $\operatorname{dom} I_1 = \operatorname{Ball}(b, r_2)$ and $\operatorname{rng} I_1 \subseteq \operatorname{Ball}(a, r_1)$ and I_1 is continuous on $\operatorname{Ball}(b, r_2)$

and $I_1(b) = a$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f_{/I_{1/y}} = y$ and I_1 is differentiable on $\text{Ball}(b, r_2)$ and $I_1{}'_{\restriction \text{Ball}(b,r_2)}$ is continuous on $\text{Ball}(b, r_2)$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $I_1{}'(y) = \text{Inv}\, f'(I_{1/y})$ and for every point y of F such that $y \in \text{Ball}(b, r_2)$ holds $f'(I_{1/y})$ is invertible. Set $B = \text{Ball}(b, r_2)$. Set $A = \text{Ball}(a, r_1) \cap f^{-1}(B)$. For every object s such that $s \in B$ holds $s \in f^{\circ}\, \text{Ball}(a, r_1)$. $f^{-1}(B)$ is open. For every object s, $s \in f^{\circ} A$ iff $s \in B$.

For every objects y_1, y_2 such that $y_1, y_2 \in \text{dom}\, I_1$ and $I_1(y_1) = I_1(y_2)$ holds $y_1 = y_2$. For every objects x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \restriction A)$ and $(f \restriction A)(x_1) = (f \restriction A)(x_2)$ holds $x_1 = x_2$. For every object x such that $x \in \text{dom}((f \restriction A)^{-1})$ holds $((f \restriction A)^{-1})(x) = I_1(x)$. □

(18) Let us consider non trivial real Banach spaces E, F, a subset Z of E, a partial function f from E to F, a point a of E, and a point b of F. Suppose Z is open and $\text{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible. Let us consider a real number r_1. Suppose $0 < r_1$. Then there exists a real number r_2 such that

  (i) $0 < r_2$, and

  (ii) $\text{Ball}(b, r_2) \subseteq f^{\circ}\, \text{Ball}(a, r_1)$.

The theorem is a consequence of (17) and (1).

(19) Let us consider non trivial real Banach spaces E, F, a subset Z of E, and a partial function f from E to F. Suppose Z is open and $\text{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and for every point x of E such that $x \in Z$ holds $f'(x)$ is invertible. Then

  (i) for every point x of E and for every real number r_1 such that $x \in Z$ and $0 < r_1$ there exists a point y of F and there exists a real number r_2 such that $y = f(x)$ and $0 < r_2$ and $\text{Ball}(y, r_2) \subseteq f^{\circ}\, \text{Ball}(x, r_1)$, and

  (ii) $f^{\circ} Z$ is open.

PROOF: For every point x of E and for every real number r_1 such that $x \in Z$ and $0 < r_1$ there exists a point y of F and there exists a real number r_2 such that $y = f(x)$ and $0 < r_2$ and $\text{Ball}(y, r_2) \subseteq f^{\circ}\, \text{Ball}(x, r_1)$. For every point y of F such that $y \in f^{\circ} Z$ there exists a real number r such that $0 < r$ and $\text{Ball}(y, r) \subseteq f^{\circ} Z$ by [7, (20)]. □

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