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Functional Sequence in Norm Space

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Summary. In this article, we formalize in Mizar [1], [2] functional sequences and basic operations on functional sequences in norm space based on [5]. In the first section, we define functional sequence in norm space. In the second section, we define pointwise convergence and prove some related theorems. In the last section we define uniform convergence and limit of functional sequence.

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1. Preliminaries

From now on D denotes a non empty set, D_1, D_2, x, y, Z denote sets, n, k denote natural numbers, p, x_1, r denote real numbers, f denotes a function, Y denotes a real normed space, and G, H, H_1, H_2, J denote sequences of partial functions from D into the carrier of Y.

Now we state the proposition:

(1) f is a sequence of partial functions from D_1 into D_2 if and only if $\operatorname{dom} f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is a partial function from D_1 to D_2.

Proof: If f is a sequence of partial functions from D_1 into D_2, then $\operatorname{dom} f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is a partial function from D_1 to D_2 by [3, (46)]. $\square$

Let us consider D. Let Y be a non empty normed structure, H be a sequence of partial functions from D into the carrier of Y, and r be a real number. The functor $r \cdot H$ yielding a sequence of partial functions from D into the carrier of Y is defined by

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(Def. 1) for every natural number n, $it(n) = r \cdot H(n)$.

Let Y be a real normed space. The functor $-H$ yielding a sequence of partial functions from D into the carrier of Y is defined by

(Def. 2) for every natural number n, $it(n) = -H(n)$.

One can verify that the functor is involutive.

Let Y be a non empty normed structure. The functor $\|H\|$ yielding a sequence of partial functions from D into $\mathbb{R}$ is defined by

(Def. 3) for every natural number n, $it(n) = \|H(n)\|$.

Let G, H be sequences of partial functions from D into the carrier of Y. The functor $G + H$ yielding a sequence of partial functions from D into the carrier of Y is defined by

(Def. 4) for every natural number n, $it(n) = G(n) + H(n)$.

Let Y be a real normed space. The functor $G - H$ yielding a sequence of partial functions from D into the carrier of Y is defined by the term

(Def. 5) $G + -H$.

Now we state the propositions:

(2) $H_1 = G - H$ if and only if for every n, $H_1(n) = G(n) - H(n)$.

PROOF: If $H_1 = G - H$, then for every n, $H_1(n) = G(n) - H(n)$ by [7, (25)]. $\square$

(3) (i) $G + H = H + G$, and

(ii) $(G + H) + J = G + (H + J)$.

(4) $-H = (-1) \cdot H$.

(5) (i) $r \cdot (G + H) = r \cdot G + r \cdot H$, and

(ii) $r \cdot (G - H) = r \cdot G - r \cdot H$.

The theorem is a consequence of (2).

(6) $r \cdot p \cdot H = r \cdot (p \cdot H)$.

(7) $1 \cdot H = H$.

(8) $\|r \cdot H\| = |r| \cdot \|H\|$.

2. Pointwise Convergence

In the sequel x denotes an element of D, X denotes a set, S_1, S_2 denote sequences of Y, and f denotes a partial function from D to the carrier of Y.

Let us consider D. Let Y be a non empty normed structure and H be a sequence of partial functions from D into the carrier of Y. Let us consider x. The functor $H\#x$ yielding a sequence of the carrier of Y is defined by

(Def. 6) for every n, $it(n) = H(n)_{/x}$.

Let us consider Y, H, and X. We say that H is point-convergent on X if and only if

(Def. 7) X is common for elements of H and there exists f such that $X = \mathrm{dom}\, f$ and for every x such that $x \in X$ for every p such that $p > 0$ there exists k such that for every n such that $n \geqslant k$ holds $\|H(n)_{/x} - f_{/x}\| < p$.

Now we state the propositions:

(9) H is point-convergent on X if and only if X is common for elements of H and there exists f such that $X = \mathrm{dom}\, f$ and for every x such that $x \in X$ holds $H\#x$ is convergent and $\lim(H\#x) = f(x)$.

(10) H is point-convergent on X if and only if X is common for elements of H and for every x such that $x \in X$ holds $H\#x$ is convergent.
Proof: Define $\mathcal{X}[\text{set}] \equiv \$_1 \in X$. Define $\mathcal{U}(\text{element of } D) = (\lim(H\#\$_1))(\in$ (the carrier of Y)). Consider f such that for every x, $x \in \mathrm{dom}\, f$ iff $\mathcal{X}[x]$ and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x) = \mathcal{U}(x)$ from [4, Sch. 3]. If H is point-convergent on X, then X is common for elements of H and for every x such that $x \in X$ holds $H\#x$ is convergent. □

3. Uniform Convergence and Limit of Functional Sequence

Let us consider D, Y, H, and X. We say that H is uniform-convergent on X if and only if

(Def. 8) X is common for elements of H and there exists f such that $X = \mathrm{dom}\, f$ and for every p such that $p > 0$ there exists k such that for every n and x such that $n \geqslant k$ and $x \in X$ holds $\|H(n)_{/x} - f_{/x}\| < p$.

Assume H is point-convergent on X. The functor $\lim_X H$ yielding a partial function from D to the carrier of Y is defined by

(Def. 9) $\mathrm{dom}\, it = X$ and for every x such that $x \in \mathrm{dom}\, it$ holds $it(x) = \lim(H\#x)$.

Now we state the propositions:

(11) Suppose H is point-convergent on X. Then $f = \lim_X H$ if and only if $\mathrm{dom}\, f = X$ and for every x such that $x \in X$ for every p such that $p > 0$ there exists k such that for every n such that $n \geqslant k$ holds $\|H(n)_{/x} - f_{/x}\| < p$. The theorem is a consequence of (10).

(12) If H is uniform-convergent on X, then H is point-convergent on X.

(13) If $Z \subseteq X$ and $Z \neq \emptyset$ and X is common for elements of H, then Z is common for elements of H.

(14) Suppose $Z \subseteq X$ and $Z \neq \emptyset$ and H is point-convergent on X. Then

(i) H is point-convergent on Z, and

(ii) $\lim_X H{\upharpoonright}Z = \lim_Z H$.

The theorem is a consequence of (13).

(15) If $Z \subseteq X$ and $Z \neq \emptyset$ and H is uniform-convergent on X, then H is uniform-convergent on Z. The theorem is a consequence of (13).

Let us consider a set x. Now we state the propositions:

(16) If X is common for elements of H, then if $x \in X$, then $\{x\}$ is common for elements of H.

(17) If H is point-convergent on X, then if $x \in X$, then $\{x\}$ is common for elements of H.

(18) Suppose $\{x\}$ is common for elements of H_1 and common for elements of H_2. Then

(i) $H_1\#x + H_2\#x = (H_1 + H_2)\#x$, and

(ii) $H_1\#x - H_2\#x = (H_1 - H_2)\#x$.

The theorem is a consequence of (2).

In the sequel x denotes an element of D.

(19) Suppose $\{x\}$ is common for elements of H. Then

(i) $\|H\|\#x = \|H\#x\|$, and

(ii) $(-H)\#x = (-1) \cdot (H\#x)$.

(20) If $\{x\}$ is common for elements of H, then $(r \cdot H)\#x = r \cdot (H\#x)$.

(21) Suppose X is common for elements of H_1 and common for elements of H_2. If $x \in X$, then $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$. The theorem is a consequence of (16) and (18).

(22) Suppose $\{x\}$ is common for elements of H. Then

(i) $\|H\|\#x = \|H\#x\|$, and

(ii) $(-H)\#x = (-1) \cdot (H\#x)$.

Let us consider x. Now we state the propositions:

(23) If X is common for elements of H, then if $x \in X$, then $(r \cdot H)\#x = r \cdot (H\#x)$. The theorem is a consequence of (16) and (20).

(24) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then if $x \in X$, then $H_1\#x + H_2\#x = (H_1 + H_2)\#x$ and $H_1\#x - H_2\#x = (H_1 - H_2)\#x$.

(25) Suppose $\{x\}$ is common for elements of H. Then

(i) $\|H\|\#x = \|H\#x\|$, and

(ii) $(-H)\#x = (-1) \cdot (H\#x)$.

(26) If H is point-convergent on X, then for every x such that $x \in X$ holds $(r \cdot H)\#x = r \cdot (H\#x)$.

(27) If X is common for elements of H_1 and common for elements of H_2, then X is common for elements of $H_1 + H_2$ and common for elements of $H_1 - H_2$. The theorem is a consequence of (2).

(28) If X is common for elements of H, then X is common for elements of $\|H\|$ and common for elements of $-H$.

(29) If X is common for elements of H, then X is common for elements of $r \cdot H$.

(30) Suppose H_1 is point-convergent on X and H_2 is point-convergent on X. Then

(i) $H_1 + H_2$ is point-convergent on X, and

(ii) $\lim_X(H_1 + H_2) = \lim_X H_1 + \lim_X H_2$, and

(iii) $H_1 - H_2$ is point-convergent on X, and

(iv) $\lim_X(H_1 - H_2) = \lim_X H_1 - \lim_X H_2$.

The theorem is a consequence of (10), (21), and (27).

(31) Suppose H is point-convergent on X. Then

(i) $\|H\|$ is point-convergent on X, and

(ii) $\lim_X\|H\| = \|\lim_X H\|$, and

(iii) $-H$ is point-convergent on X, and

(iv) $\lim_X(-H) = -\lim_X H$.

The theorem is a consequence of (16), (10), (19), and (28).

(32) If H is point-convergent on X, then $r \cdot H$ is point-convergent on X and $\lim_X(r \cdot H) = r \cdot \lim_X H$. The theorem is a consequence of (10), (23), and (29).

(33) H is uniform-convergent on X if and only if X is common for elements of H and H is point-convergent on X and for every r such that $0 < r$ there exists k such that for every n and x such that $n \geqslant k$ and $x \in X$ holds $\|H(n)_{/x} - (\lim_X H)_{/x}\| < r$. The theorem is a consequence of (12) and (11).

From now on V, W denote real normed spaces and H denotes a sequence of partial functions from the carrier of V into the carrier of W.

Now we state the proposition:

(34) If H is uniform-convergent on X and for every n, $H(n)\restriction X$ is continuous on X, then $\lim_X H$ is continuous on X.

Proof: Set $l = \lim_X H$. H is point-convergent on X. For every point x_0 of V such that $x_0 \in X$ holds $l{\upharpoonright}X$ is continuous in x_0 by [6, (62)], (33), (11), [6, (61)]. $\square$

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