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Extended Natural Numbers and Counters

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Summary. This article introduces extended natural numbers, i.e. the set $\mathbb{N} \cup \{+\infty\}$, in Mizar [4], [3] and formalizes a way to list a cardinal numbers of cardinals. Both concepts have applications in graph theory.

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0. Introduction

Extended natural numbers have often been used in the literature to define distances in graphs that are not necessarily connected, to set the distance between vertices of different components to $+\infty$, see e.g. [5], [7], [8]. Therefore it is only natural to formalize these numbers in preparation for a formalization of distances in graphs. On the other hand, one usually does not see the list of counters from the second part of this article in the literature. The generalistic motivation to introduce these is a rather simple one, however. n-partite finite graphs are rather known and constructions like $K_{\omega,\omega}$ arise sometimes. The index objects of these alone could be formalized using `Cardinal-yielding XFinSequence` (cf. [14], [1]), but a generalization for the index object to be any cardinality long seemed to be appropriate. This allows for easy notation of more graphs than just with the finite amount of indices. For example $K_{1,2,3,\ldots}$, where the index ranges over all natural numbers, is an easy notation for a graph that does not

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have a finite independence number and also no infinite subset of vertices that form an independent set.

In the first section the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ of extended natural numbers is introduced to the Mizar system [6] as a subset of the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ defined in [12]. Basic theorems will be proven, often specializations of theorems from [10], [13] or generalizations of theorems from [2]. The second section will introduce sets of extended natural numbers and proceed in a similar fashion to [11]. The third section does the same with relations that only have extended natural numbers in their range, similar to [9]. Section 4 deals with some ordinal preliminaries. Not all are needed for the last section, but the author felt they would fit better here than into a graph preliminary article. Finally, the last section introduces relations with cardinal domain, as only a cardinal domain (in lieu of an ordinal one) is needed for counting purposes. The article ends with the definition of `Counters` and `Counters+`, two expandable modes with the latter not allowing 0 in its range.

1. Extended Natural Numbers

The functor $\overline{\mathbb{N}}$ yielding a subset of $\overline{\mathbb{R}}$ is defined by the term

(Def. 1) $\mathbb{N} \cup \{+\infty\}$.

Now we state the proposition:

(1) $\mathbb{N} \subset \overline{\mathbb{N}} \subset \overline{\mathbb{R}}$.

Proof: $-\infty \notin \overline{\mathbb{N}}$. □

Observe that $\overline{\mathbb{N}}$ is non empty and infinite.

Let x be an object. We say that x is extended natural if and only if

(Def. 2) $x \in \overline{\mathbb{N}}$.

Let us observe that $+\infty$ is extended natural and every object which is extended natural is also extended real and every object which is natural is also extended natural and every set which is finite and extended natural is also natural.

There exists an object which is zero and extended natural and there exists an object which is non zero and extended natural and there exists a number which is extended natural and every element of $\overline{\mathbb{N}}$ is extended natural.

An extended natural is an extended natural extended real. Let x be an extended natural. Note that $x (\in \overline{\mathbb{N}})$ reduces to x.

One can check that sethood property holds for extended naturals.

Now we state the proposition:

(2) Let us consider an object x. Then x is an extended natural if and only if x is a natural number or $x = +\infty$.

Note that every object which is zero is also extended natural and every extended real which is extended natural is also non negative and every extended natural is non negative and every extended natural which is non zero is also positive.

From now on N, M, K denote extended naturals.

Let us consider N and M. Observe that $\min(N, M)$ is extended natural and $\max(N, M)$ is extended natural and $N + M$ is extended natural and $N \cdot M$ is extended natural.

Now we state the propositions:

(3) $0 \leqslant N$.

(4) If $0 \neq N$, then $0 < N$.

(5) $0 < N + 1$.

(6) If $M \in \mathbb{N}$ and $N \leqslant M$, then $N \in \mathbb{N}$.

(7) If $N < M$, then $N \in \mathbb{N}$.

(8) If $N \leqslant M$, then $N \cdot K \leqslant M \cdot K$.

(9) (i) $N = 0$, or

(ii) there exists K such that $N = K + 1$.

The theorem is a consequence of (2).

(10) If $N + M = 0$, then $N = 0$ and $M = 0$.

Let M be an extended natural and N be a non zero extended natural. One can check that $M + N$ is non zero and $N + M$ is non zero.

Now we state the propositions:

(11) If $N \leqslant M + 1$, then $N \leqslant M$ or $N = M + 1$.

(12) If $N \leqslant M \leqslant N + 1$, then $N = M$ or $M = N + 1$.

(13) If $N \leqslant M$, then there exists K such that $M = N + K$.

(14) $N \leqslant N + M$.

(15) If $N \leqslant M$, then $N \leqslant M + K$.

(16) If $N < 1$, then $N = 0$.

(17) If $N \cdot M = 1$, then $N = 1$.

(18) $K < K + N$ if and only if $1 \leqslant N$ and $K \neq +\infty$.

(19) If $K \neq 0$ and $N = M \cdot K$, then $M \leqslant N$.

(20) If $M \leqslant N$, then $M \cdot K \leqslant N \cdot K$.

(21) $(K + M) + N = K + (M + N)$.

(22) $K \cdot (N + M) = K \cdot N + K \cdot M$.

2. Sets of Extended Natural Numbers

Let X be a set. We say that X is extended natural-membered if and only if

(Def. 3) for every object x such that $x \in X$ holds x is extended natural.

Note that every set which is empty is also extended natural-membered and every set which is natural-membered is also extended natural-membered.

Every set which is extended natural-membered is also extended real-membered and $\overline{\mathbb{N}}$ is extended natural-membered and there exists a set which is non empty and extended natural-membered. Now we state the proposition:

(23) Let us consider a set X. Then X is extended natural-membered if and only if $X \subseteq \overline{\mathbb{N}}$.

In the sequel X denotes an extended natural-membered set.

Let us consider X. Let us observe that every element of X is extended natural. Now we state the propositions:

(24) Let us consider a non empty, extended natural-membered set X. Then there exists N such that $N \in X$.

(25) If for every N, $N \in X$, then $X = \overline{\mathbb{N}}$.

(26) Let us consider a set Y. If $Y \subseteq X$, then Y is extended natural-membered.

Let us consider X. One can verify that every subset of X is extended natural-membered. Let us consider N. Let us observe that $\{N\}$ is extended natural-membered. Let us consider M. Let us note that $\{N, M\}$ is extended natural-membered. Let us consider K. One can verify that $\{N, M, K\}$ is extended natural-membered.

Let us consider X. Let Y be an extended natural-membered set. One can verify that $X \cup Y$ is extended natural-membered.

Let Y be a set. One can verify that $X \cap Y$ is extended natural-membered and $X \setminus Y$ is extended natural-membered.

Let Y be an extended natural-membered set. One can check that $X \dot{-} Y$ is extended natural-membered.

Let Y be a set. One can check that $X \subseteq Y$ if and only if the condition (Def. 4) is satisfied.

(Def. 4) if $N \in X$, then $N \in Y$.

Let Y be an extended natural-membered set. One can check that $X = Y$ if and only if the condition (Def. 5) is satisfied.

(Def. 5) $N \in X$ iff $N \in Y$.

One can verify that X misses Y if and only if the condition (Def. 6) is satisfied.

(Def. 6) there exists no N such that $N \in X$ and $N \in Y$.

Now we state the propositions:

(27) Let us consider a set F. Suppose for every set X such that $X \in F$ holds X is extended natural-membered. Then $\bigcup F$ is extended natural-membered.

(28) Let us consider sets F, X. Suppose $X \in F$ and X is extended natural-membered. Then $\bigcap F$ is extended natural-membered.

The scheme *ENMSeparation* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists an extended natural-membered set X such that for every N, $N \in X$ iff $\mathcal{P}[N]$.

Let X be an extended natural-membered set. Let us note that an upper bound of X can equivalently be formulated as follows:

(Def. 7) for every N such that $N \in X$ holds $N \leqslant it$.

One can check that a lower bound of X can equivalently be formulated as follows:

(Def. 8) for every N such that $N \in X$ holds $it \leqslant N$.

Let us note that every extended natural-membered set is lower bounded and every extended natural-membered set which is non empty is also left-ended.

Let us consider X. Note that there exists an upper bound of X which is extended natural and there exists a lower bound of X which is extended natural and $\inf X$ is extended natural.

Let X be a non empty, extended natural-membered set. Let us note that $\sup X$ is extended natural and every extended natural-membered set which is non empty and upper bounded is also right-ended.

Let X be a left-ended, extended natural-membered set. One can verify that the functor $\min X$ yields an extended natural and is defined by

(Def. 9) $it \in X$ and for every N such that $N \in X$ holds $it \leqslant N$.

Let X be a right-ended, extended natural-membered set. One can verify that the functor $\max X$ yields an extended natural and is defined by

(Def. 10) $it \in X$ and for every N such that $N \in X$ holds $N \leqslant it$.

3. Relations with Extended Natural Numbers in Range

Let R be a binary relation. We say that R is extended natural-valued if and only if

(Def. 11) $\operatorname{rng} R \subseteq \overline{\mathbb{N}}$.

Let us note that every binary relation which is empty is also extended natural-valued and every binary relation which is natural-valued is also extended natural-valued and every binary relation which is extended natural-valued is also ($\overline{\mathbb{N}}$)-valued and extended real-valued.

Every binary relation which is $(\overline{\mathbb{N}})$-valued is also extended natural-valued and there exists a function which is extended natural-valued.

Let R be an extended natural-valued binary relation. One can check that $\operatorname{rng} R$ is extended natural-membered.

Now we state the proposition:

(29) Let us consider a binary relation R, and an extended natural-valued binary relation S. If $R \subseteq S$, then R is extended natural-valued.

Let R be an extended natural-valued binary relation. Observe that every subset of R is extended natural-valued.

Let R, S be extended natural-valued binary relations. One can verify that $R \cup S$ is extended natural-valued.

Let R be an extended natural-valued binary relation and S be a binary relation. One can check that $R \cap S$ is extended natural-valued and $R \setminus S$ is extended natural-valued and $S \cdot R$ is extended natural-valued.

Let R, S be extended natural-valued binary relations. Note that $R \dot{-} S$ is extended natural-valued.

Let R be an extended natural-valued binary relation and X be a set. Let us note that $R^{\circ}X$ is extended natural-membered and $R{\upharpoonright}X$ is extended natural-valued and $X{\restriction}R$ is extended natural-valued.

Let x be an object. Let us observe that $R^{\circ}x$ is extended natural-membered.

Let us consider X. One can check that id_X is extended natural-valued.

Let f be a function. Note that f is extended natural-valued if and only if the condition (Def. 12) is satisfied.

(Def. 12) for every object x such that $x \in \operatorname{dom} f$ holds $f(x)$ is extended natural.

Now we state the proposition:

(30) Let us consider a function f. Then f is extended natural-valued if and only if for every object x, $f(x)$ is extended natural.

Let f be an extended natural-valued function and x be an object. Observe that $f(x)$ is extended natural.

Let X be a set. Let us consider N. One can verify that $X \longmapsto N$ is extended natural-valued.

Let f, g be extended natural-valued functions. Note that $f + \cdot g$ is extended natural-valued.

Let x be an object. Let us consider N. Let us observe that $x \dot{\longmapsto} N$ is extended natural-valued.

Let Z be a set. Let us consider X. Note that every relation between Z and X is extended natural-valued and $Z \times X$ is extended natural-valued as a relation between Z and X and there exists a function which is non empty, constant, and extended natural-valued.

Now we state the proposition:

(31) Let us consider a non empty, constant, extended natural-valued function f. Then there exists N such that for every object x such that $x \in \operatorname{dom} f$ holds $f(x) = N$.

4. Ordinal Preliminaries

Now we state the proposition:

(32) Let us consider a function f. Then f is ordinal yielding if and only if for every object x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an ordinal number.

One can check that every set which is ordinal is also $\subseteq$-linear.

Let f be an ordinal yielding function and x be an object. Observe that $f(x)$ is ordinal.

Let A, B be non-empty transfinite sequences. Note that $A \frown B$ is non-empty.

Now we state the propositions:

(33) Let us consider a set X, and an object x. Then $\overline{\overline{X \longmapsto x}} = \overline{\overline{X}}$.

(34) Let us consider a cardinal number c, and an object x. Then $\overline{\overline{c \longmapsto x}} = c$. The theorem is a consequence of (33).

Let X be a trivial set. One can verify that $\overline{\overline{X}}$ is trivial.

Let c_1 be a cardinal number and c_2 be a non empty cardinal number. Note that $c_1 + c_2$ is non empty.

Now we state the propositions:

(35) Let us consider an ordinal number A. Then $A \neq 0$ and $A \neq 1$ if and only if A is not trivial.

(36) Let us consider an ordinal number A, and an infinite cardinal number B. If $A \in B$, then $A + B = B$.
Proof: Define $\mathcal{F}(\text{ordinal number}) = A + \$_1$. Consider f being a sequence of ordinal numbers such that $\operatorname{dom} f = B$ and for every ordinal number C such that $C \in B$ holds $f(C) = \mathcal{F}(C)$. □

Let f be a cardinal yielding function and g be a function. Observe that $f \cdot g$ is cardinal yielding and every function which is natural-valued is also cardinal yielding.

Let f be an empty function. Let us observe that disjoint f is empty.

Let f be an empty yielding function. One can verify that disjoint f is empty yielding.

Let f be a non empty yielding function. One can check that disjoint f is non empty yielding.

Let f be an empty yielding function. One can verify that $\bigcup f$ is empty and every function which is cardinal yielding is also ordinal yielding.

Now we state the proposition:

(37) Let us consider a function f, and a permutation p of $\operatorname{dom} f$. Then $\operatorname{Card}(f \cdot p) = (\operatorname{Card} f) \cdot p$.

Let A be a transfinite sequence. Note that $\operatorname{Card} A$ is transfinite sequence-like.

Now we state the proposition:

(38) Let us consider transfinite sequences A, B. Then $\operatorname{Card}(A ^\frown B) = \operatorname{Card} A ^\frown \operatorname{Card} B$.

Let f be a trivial function. One can check that $\operatorname{Card} f$ is trivial.

Let f be a non trivial function. Note that $\operatorname{Card} f$ is non trivial.

Let A, B be cardinal yielding transfinite sequences. Note that $A ^\frown B$ is cardinal yielding.

Let c_1 be a cardinal number. Note that $\langle c_1 \rangle$ is cardinal yielding.

Let c_2 be a cardinal number. Let us observe that $\langle c_1, c_2 \rangle$ is cardinal yielding.

Let c_3 be a cardinal number. One can verify that $\langle c_1, c_2, c_3 \rangle$ is cardinal yielding.

Let X_1, X_2, X_3 be non empty sets. One can verify that $\langle X_1, X_2, X_3 \rangle$ is non-empty.

Let A be an infinite ordinal number. Let us note that $\langle A \rangle$ is non natural-valued.

Let x be an object. Let us observe that $\langle A, x \rangle$ is non natural-valued and $\langle x, A \rangle$ is non natural-valued.

Let y be an object. Observe that $\langle A, x, y \rangle$ is non natural-valued and $\langle x, A, y \rangle$ is non natural-valued and $\langle x, y, A \rangle$ is non natural-valued and there exists a finite 0-sequence which is non empty, non-empty, and natural-valued and $\langle x \rangle$ is one-to-one.

Now we state the propositions:

(39) Let us consider objects x, y. Then

(i) $\operatorname{dom}\langle x, y \rangle = \{0, 1\}$, and

(ii) $\operatorname{rng}\langle x, y \rangle = \{x, y\}$.

(40) Let us consider objects x, y, z. Then

(i) $\operatorname{dom}\langle x, y, z \rangle = \{0, 1, 2\}$, and

(ii) $\operatorname{rng}\langle x, y, z \rangle = \{x, y, z\}$.

Let x be an object. One can verify that $\langle x \rangle$ is trivial.

Let y be an object. Let us note that $\langle x, y \rangle$ is non trivial.

Let z be an object. Let us note that $\langle x, y, z \rangle$ is non trivial and there exists a finite 0-sequence which is non empty and trivial.

Let D be a non empty set. One can check that there exists a finite 0-sequence of D which is non empty and trivial.

Now we state the propositions:

(41) Let us consider a non empty, trivial transfinite sequence p. Then there exists an object x such that $p = \langle x \rangle$.

(42) Let us consider a non empty set D, and a non empty, trivial transfinite sequence p of elements of D. Then there exists an element x of D such that $p = \langle x \rangle$. The theorem is a consequence of (41).

(43) $\langle 0 \rangle = \mathrm{id}_1$.

(44) $\langle 0, 1 \rangle = \mathrm{id}_2$. The theorem is a consequence of (39).

(45) $\langle 0, 1, 2 \rangle = \mathrm{id}_3$. The theorem is a consequence of (40).

(46) Let us consider objects x, y. Then $\langle x, y \rangle \cdot \langle 1, 0 \rangle = \langle y, x \rangle$. The theorem is a consequence of (39).

Let us consider objects x, y, z. Now we state the propositions:

(47) $\langle x, y, z \rangle \cdot \langle 0, 2, 1 \rangle = \langle x, z, y \rangle$. The theorem is a consequence of (40).

(48) $\langle x, y, z \rangle \cdot \langle 1, 0, 2 \rangle = \langle y, x, z \rangle$. The theorem is a consequence of (40).

(49) $\langle x, y, z \rangle \cdot \langle 1, 2, 0 \rangle = \langle y, z, x \rangle$. The theorem is a consequence of (40).

(50) $\langle x, y, z \rangle \cdot \langle 2, 0, 1 \rangle = \langle z, x, y \rangle$. The theorem is a consequence of (40).

(51) $\langle x, y, z \rangle \cdot \langle 2, 1, 0 \rangle = \langle z, y, x \rangle$. The theorem is a consequence of (40).

(52) Let us consider objects x, y. If $x \neq y$, then $\langle x, y \rangle$ is one-to-one. The theorem is a consequence of (39).

(53) Let us consider objects x, y, z. If $x \neq y$ and $x \neq z$ and $y \neq z$, then $\langle x, y, z \rangle$ is one-to-one. The theorem is a consequence of (40).

5. Relations with Cardinal Domain

Let R be a binary relation. We say that R is with cardinal domain if and only if

(Def. 13) there exists a cardinal number c such that $\mathrm{dom}\, R = c$.

One can verify that every binary relation which is empty is also with cardinal domain and every binary relation which is finite and transfinite sequence-like is also with cardinal domain and every binary relation which is with cardinal domain is also transfinite sequence-like.

Let c be a cardinal number. Let us observe that every many sorted set indexed by c is with cardinal domain.

Let x be an object. Let us note that $c \longmapsto x$ is with cardinal domain.

Let X be a set. Let us note that every denumeration of X is with cardinal domain.

Let c be a cardinal number. One can verify that every permutation of c is with cardinal domain and there exists a function which is non empty, trivial, non-empty, with cardinal domain, and cardinal yielding and there exists a function which is non empty, non trivial, non-empty, finite, with cardinal domain, and cardinal yielding.

There exists a function which is non empty, non-empty, infinite, with cardinal domain, and natural-valued and there exists a function which is non trivial, non-empty, with cardinal domain, cardinal yielding, and non natural-valued.

Let R be a with cardinal domain binary relation. One can check that $\mathrm{dom}\, R$ is cardinal.

Let f be a with cardinal domain function. We identify $\overline{\overline{f}}$ with $\mathrm{dom}\, f$. Let R be a with cardinal domain binary relation and P be a total, $(\mathrm{rng}\, R)$-defined binary relation. One can verify that $R \cdot P$ is with cardinal domain.

Let g be a function and f be a denumeration of $\mathrm{dom}\, g$. Let us observe that $g \cdot f$ is with cardinal domain.

Let f be a with cardinal domain function and p be a permutation of $\mathrm{dom}\, f$. Observe that $f \cdot p$ is with cardinal domain.

Now we state the proposition:

(54) Let us consider with cardinal domain transfinite sequences A, B. Suppose $\mathrm{dom}\, A \in \mathrm{dom}\, B$. Then $A \frown B$ is with cardinal domain. The theorem is a consequence of (36).

Let p be a finite 0-sequence and B be a with cardinal domain transfinite sequence. Observe that $p \frown B$ is with cardinal domain.

A Counters is a non empty, with cardinal domain, cardinal yielding function.

A $\mathrm{Counters}_{+}$ is a non empty, non-empty, with cardinal domain, cardinal yielding function.

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