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A Case Study of Transporting Urysohn's Lemma from Topology via Open Sets into Topology via Neighborhoods

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Summary. Józef Białas and Yatsuka Nakamura has completely formalized a proof of Urysohn's lemma in the article [4], in the context of a topological space defined via open sets. In the Mizar Mathematical Library (MML), the topological space is defined in this way by Beata Padlewska and Agata Darmochwał in the article [18]. In [7] the topological space is defined via neighborhoods. It is well known that these definitions are equivalent [5, 6].

In the definitions, an abstract structure (i.e. the article [17, `STRUCT_0`] and its descendants, all of them directly or indirectly using Mizar structures [3]) have been used (see [10], [9]). The first topological definition is based on the Mizar structure `TopStruct` and the topological space defined via neighborhoods with the Mizar structure: `FMT_Space_Str`. To emphasize the notion of a neighborhood, we rename `FMT_TopSpace` (topology from neighbourhoods) to `NTopSpace` (a neighborhood topological space).

Using Mizar [2], we transport the Urysohn's lemma from `TopSpace` to `NTopSpace`.

In some cases, Mizar allows certain techniques for transporting proofs, definitions or theorems. Generally speaking, there is no such automatic translating.

In Coq, Isabelle/HOL or homotopy type theory transport is also studied, sometimes with a more systematic aim [14], [21], [11], [12], [8], [19]. In [1], two co-existing Isabelle libraries: Isabelle/HOL and Isabelle/Mizar, have been aligned in a single foundation in the Isabelle logical framework.

In the MML, they have been used since the beginning: `reconsider`, `registration`, `cluster`, others were later implemented [13]: `identify`.

In some proofs, it is possible to define particular functors between different structures, mainly useful when results are already obtained in a given structure. This technique is used, for example, in [15] to define two functors `MXR2MXF` and `MXF2MXF` between `Matrix of REAL` and `Matrix of F-Real` and to transport the

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definition of the addition from one structure to the other: [...] `A + B -> Matrix of REAL equals MXF2MXR ((MXR2MXF A) + (MXR2MXF B))` [...].

In this paper, first we align the necessary topological concepts. For the formalization, we were inspired by the works of Claude Wagschal [20]. It allows us to transport more naturally the Urysohn's lemma ([4, `URYSOHN3:20`]) to the topological space defined via neighborhoods.

Nakasho and Shidama have developed a solution to explore the notions introduced in various ways `https://mimosa-project.github.io/mmlreference/current/` [16].

The definitions can be directly linked in the HTML version of the Mizar library (example: Urysohn's lemma `http://mizar.org/version/current/html/urysohn3.html#T20`).

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1. Some Redefinitions: Neighborhood Topological Space

From now on T denotes a topological space and A, B denote subsets of T. Now we state the proposition:

(1) If A misses B, then $\operatorname{Int} A$ misses $\operatorname{Int} B$.

A neighborhood topological space is a topology from neighbourhoods. Let X be a non empty topological space. We introduce the notation Top2NTop(X) as a synonym of TopSpace2FMT X.

Let X be a topology from neighbourhoods. We introduce the notation NTop2-Top(X) as a synonym of FMT2TopSpace X.

2. Alignment of Topological Space Concepts Defined via Open Sets and Defined via Neighbourhoods

Let N_1 be a non empty neighborhood topological space. Observe that Ω_{N_1} is open and $\emptyset_{N_1}$ is open.

Let N_1 be a U-FMT filter, non empty, strict formal topological space and x be an element of N_1. Note that the functor $U_F(x)$ yields a filter of the carrier of N_1.

[20, definition 2.11.2, p. 89]:

Let N_1 be a U-FMT filter, non empty, strict formal topological space and F be a filter of the carrier of N_1. The functor LimFilter(F) yielding a subset of N_1 is defined by the term

(Def. 1) $\{x$, where x is a point of $N_1 : F$ is finer than $U_F(x)\}$.

[20, DEFINITION 2.11.3, P. 92 AND PROPOSITION 2.11.4, P. 90]:

Let N_1, N_2 be U-FMT filter, non empty, strict formal topological spaces, f be a function from N_1 into N_2, and F be a filter of the carrier of N_1. The functor $\lim_F f$ yielding a subset of N_2 is defined by the term

(Def. 2) LimFilter(the image of filter F under f).

[20, DEFINITION 2.10.1 (1), P. 83]:

Let N be a neighborhood topological space, A be a subset of N, and x be a point of N. We say that x is interior point of A if and only if

(Def. 3) A is a neighbourhood of x.

[20, DEFINITION 2.10.1 (2), P. 83]:

Let N be a neighborhood topological space, A be a subset of N, and x be a point of N. We say that x is adherent point of A if and only if

(Def. 4) for every element V of $U_F(x)$, V meets A.

The functor $\operatorname{Int} A$ yielding a subset of N is defined by the term

(Def. 5) $\{x$, where x is a point of $N : x$ is interior point of $A\}$.

[20, DEFINITION 2.13.1, P. 97]:

Let N_1, N_2 be neighborhood topological spaces, f be a function from N_1 into N_2, and x be a point of N_1. We say that f is continuous at x if and only if

(Def. 6) for every filter F of the carrier of N_1 such that $x \in \text{LimFilter}(F)$ holds $f(x) \in \lim_F f$.

We say that f is continuous if and only if

(Def. 7) for every point x of N_1, f is continuous at x.

Note that there exists a function from N_1 into N_2 which is continuous.

Let N be a neighborhood topological space and A be a subset of N.

[20, DEFINITION 2.10.1 (1), P. 83]: $\operatorname{Int} A$ is open.

[20, DEFINITION 2.10.1 (2), P. 83]:

Let N be a neighborhood topological space and A be a subset of N. The functor $\overline{A}$ yielding a subset of N is defined by the term

(Def. 8) $\{x$, where x is a point of $N : x$ is adherent point of $A\}$.

[20, DEFINITION 2.9.3, P. 81]:

Let N_1 be a neighborhood topological space and A be a subset of N_1. We say that A is closed if and only if

(Def. 9) $\Omega_{N_1} \setminus A$ is an open subset of N_1.

One can check that there exists a subset of N_1 which is closed and Ω_{N_1} is closed as a subset of N_1 and $\emptyset_{N_1}$ is closed as a subset of N_1 and there exists a subset of N_1 which is non empty and closed.

Let S, T be non empty topological spaces and f be a function from S into T. The functor Top2NTop(f) yielding a function from Top2NTop(S) into Top2NTop(T) is defined by the term

(Def. 10) f.

Let T_1 be a non empty topological space, T_2 be a non empty, strict topological space, and f be a continuous function from T_1 into T_2. Observe that the functor Top2NTop(f) yields a continuous function from Top2NTop(T_1) into Top2NTop(T_2) and is defined by the term

(Def. 11) f.

[20, DEFINITION 2.17.1, P. 111]:

Let N be a neighborhood topological space. We say that N is T_2 if and only if

(Def. 12) for every filter F of the carrier of N, LimFilter(F) is trivial.

One can check that there exists a neighborhood topological space which is T_2.

Let N be a neighborhood topological space. We say that N is normal if and only if

(Def. 13) for every closed subsets A, B of N such that A misses B there exists a neighbourhood V of A and there exists a neighbourhood W of B such that V misses W.

Let x be a point of N. The functor NTop2Top(x) yielding a point of NTop2Top (N) is defined by the term

(Def. 14) x.

Let T be a non empty topological space and x be a point of T. The functor Top2NTop(x) yielding a point of Top2NTop(T) is defined by the term

(Def. 15) x.

Let N be a neighborhood topological space and S be a subset of N. The functor NTop2Top(S) yielding a subset of NTop2Top(N) is defined by the term

(Def. 16) S.

Let T be a non empty topological space and S be a subset of T. The functor Top2NTop(S) yielding a subset of Top2NTop(T) is defined by the term

(Def. 17) S.

One can verify that there exists a neighborhood topological space which is non empty and normal.

Let T_1, T_2 be neighborhood topological spaces and f be a function from T_1 into T_2. The functor NTop2Top(f) yielding a function from NTop2Top(T_1) into NTop2Top(T_2) is defined by the term

(Def. 18) f.

The functor FMT-$\mathbb{R}^1$ yielding a neighborhood topological space is defined by the term

(Def. 19) Top2NTop($\mathbb{R}^{\mathbf{1}}$).

Now we state the proposition:

(2) The carrier of FMT-$\mathbb{R}^1 = \mathbb{R}$.

One can verify that FMT-$\mathbb{R}^1$ is real-membered.

3. Some Properties of a Neighborhood Topology

From now on N, N_1, N_2 denote neighborhood topological spaces, A, B denote subsets of N, O denotes an open subset of N, a denotes a point of N, X denotes a subset of N_1, Y denotes a subset of N_2, x denotes a point of N_1, y denotes a point of N_2, f denotes a function from N_1 into N_2, and f_1 denotes a continuous function from N_1 into N_2.

Now we state the propositions:

(3) O is an open subset of NTop2Top(N).

(4) A is a subset of NTop2Top(N).

(5) (i) Ω_N is open, and

(ii) $\emptyset_N$ is open.

(6) $N \longmapsto y$ is continuous.

(7) a is interior point of A if and only if there exists an open subset O of N such that $a \in O$ and $O \subseteq A$.

(8) If $a \in O$, then a is interior point of O.

(9) $\operatorname{Int} A = \bigcup\{O$, where O is an open subset of $N : O \subseteq A\}$.

(10) $\operatorname{Int} A \subseteq A$.

(11) [20, DEFINITION 2.10.1, P. 83]:
If $A \subseteq B$, then $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(12) [20, DEFINITION 2.10.2, P. 83]:
A is open if and only if $\operatorname{Int} A = A$.

(13) $\operatorname{Int} A = \operatorname{Int} \operatorname{Int} A$.

(14) Let us consider a non empty, strict neighborhood topological space N, a subset A of N, and a point x of N. Suppose A is a neighbourhood of x. Then $\operatorname{Int} A$ is an open neighbourhood of x. The theorem is a consequence of (12).

(15) The image of filter $U_F(x)$ under $f = \{M$, where M is a subset of $N_2 :$ $f^{-1}(M) \in U_F(x)\}$.

(16) If f is continuous at x and $y = f(x)$, then for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$.

(17) If $y = f(x)$ and for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$, then f is continuous at x.

(18) [20, DEFINITION 2.13.1, P. 97]:
If $y = f(x)$, then f is continuous at x iff for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$.

(19) [20, PROPOSITION 2.13.3, P. 99]:
If f is continuous at x and x is adherent point of X and $y = f(x)$ and $Y = f^\circ X$, then y is adherent point of Y.

(20) [20, THEOREM 2.13.4, P. 99, (1) $\Rightarrow$ (2)]:
$f_1{}^\circ\overline{X} \subseteq \overline{f_1{}^\circ X}$.

(21) Every closed subset of N is a closed subset of NTop2Top(N).

(22) [20, PROPOSITION 2.10.2, P. 84]:
If $B = \Omega_N \setminus A$, then $\Omega_N \setminus \overline{A} = \operatorname{Int} B$.

(23) [20, PROPOSITION 2.10.2, P. 84]:
If $B = \Omega_N \setminus A$, then $\Omega_N \setminus (\operatorname{Int} A) = \overline{B}$.

(24) $A \subseteq \overline{A}$.

(25) [20, 2.10.6, P. 84]:
A is closed if and only if $\overline{A} = A$.

(26) [20, 2.10.5, P.84]:
If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(27) [20, THEOREM 2.13.4, P. 99, (2) $\Rightarrow$ (3)]:
If for every subset X of N_1, $f^\circ\overline{X} \subseteq \overline{f^\circ X}$, then for every closed subset S of N_2, $f^{-1}(S)$ is a closed subset of N_1.

(28) [20, DEFINITION 2.9.3, P. 81]:
If $B = \Omega_N \setminus A$, then A is open iff B is closed.

(29) If $A = \Omega_N \setminus B$, then A is open iff B is closed.

(30) [20, THEOREM 2.13.4, P. 99, (3) $\Rightarrow$ (4)]:
If for every closed subset S of N_2, $f^{-1}(S)$ is a closed subset of N_1, then for every open subset S of N_2, $f^{-1}(S)$ is an open subset of N_1.

(31) [20, THEOREM 2.13.4, P. 99, (4) $\Rightarrow$ (1)]:
If for every open subset S of N_2, $f^{-1}(S)$ is an open subset of N_1, then f is continuous.

(32) [20, THEOREM 2.13.4, P. 99, (1) $\Leftrightarrow$ (4)]:
f is continuous if and only if for every open subset O of N_2, $f^{-1}(O)$ is an open subset of N_1.

(33) [20, theorem 2.13.4, p. 99, (1) ⇔ (3)]:
f is continuous if and only if for every closed subset O of N_2, $f^{-1}(O)$ is a closed subset of N_1.

(34) $\mathrm{Int}\,A = \mathrm{Int}\,\mathrm{NTop2Top}(A)$.

(35) If A is a neighbourhood of a, then $\mathrm{NTop2Top}(A)$ is a neighbourhood of $\mathrm{NTop2Top}(a)$. The theorem is a consequence of (34).

(36) If A is a neighbourhood of B, then $\mathrm{NTop2Top}(A)$ is a neighbourhood of $\mathrm{NTop2Top}(B)$.

(37) If A misses B, then $\mathrm{NTop2Top}(A)$ misses $\mathrm{NTop2Top}(B)$.

(38) If A misses B, then $\mathrm{Int}\,A$ misses $\mathrm{Int}\,B$.

From now on N denotes a T_2 neighborhood topological space.

Now we state the propositions:

(39) Let us consider points x, y of N. Suppose $x \neq y$. Then there exists an element V_1 of $U_F(x)$ and there exists an element V_2 of $U_F(y)$ such that V_1 misses V_2.

(40) $\mathrm{NTop2Top}(N)$ is a T_2, non empty, strict topological space. The theorem is a consequence of (39).

(41) Let us consider a non empty, normal neighborhood topological space N. Then $\mathrm{NTop2Top}(N)$ is normal. The theorem is a consequence of (36) and (1).

Let N be a non empty, normal neighborhood topological space. One can verify that $\mathrm{NTop2Top}(N)$ is normal.

4. Some Connections between Neighborhood Topology and Open-Set Topology

In the sequel T denotes a non empty topological space, A, B denote subsets of T, F denotes a closed subset of T, and O denotes an open subset of T.

Now we state the propositions:

(42) A is a subset of $\mathrm{Top2NTop}(T)$.

(43) F is a closed subset of $\mathrm{Top2NTop}(T)$.

(44) O is an open subset of $\mathrm{Top2NTop}(T)$.

(45) If A misses B, then $\mathrm{Top2NTop}(A)$ misses $\mathrm{Top2NTop}(B)$.

(46) Let us consider a T_2, non empty topological space T. Then $\mathrm{Top2NTop}(T)$ is a T_2 neighborhood topological space.

In the sequel T denotes a non empty, strict topological space, A, B denote subsets of T, and x denotes a point of T.

Now we state the propositions:

(47) $\operatorname{Int} A = \operatorname{Int} \operatorname{Top2NTop}(A)$.

(48) If A is a neighbourhood of B, then $\operatorname{Top2NTop}(A)$ is a neighbourhood of $\operatorname{Top2NTop}(B)$.

(49) If A is a neighbourhood of x, then $\operatorname{Top2NTop}(A)$ is a neighbourhood of $\operatorname{Top2NTop}(x)$.

(50) Let us consider a non empty, normal, strict topological space T. Then $\operatorname{Top2NTop}(T)$ is normal.

Let T be a non empty, normal, strict topological space. Note that Top2NTop (T) is normal.

5. Transport from $\mathbb{R}^1$ to FMT-$\mathbb{R}^1$

From now on A denotes a subset of FMT-$\mathbb{R}^1$, x denotes a point of FMT-$\mathbb{R}^1$, y denotes a point of the metric space of real numbers, z denotes a point of (the metric space of real numbers)$_{\mathrm{top}}$, and r denotes a real number.

Now we state the propositions:

(51) $\operatorname{NTop2Top}(\text{FMT-}\mathbb{R}^1) = \mathbb{R}^{\mathbf{1}}$.

(52) The carrier of FMT-$\mathbb{R}^1 = \mathbb{R}$.

(53) Let us consider a neighborhood topological space N, and a function f from N into FMT-$\mathbb{R}^1$. Then $\operatorname{NTop2Top}(f)$ is a function from $\operatorname{NTop2Top}(N)$ into $\mathbb{R}^{\mathbf{1}}$.

(54) Let us consider a non empty topological space T, and a function f from T into $\mathbb{R}^{\mathbf{1}}$. Then $\operatorname{Top2NTop}(f)$ is a function from $\operatorname{Top2NTop}(T)$ into $\operatorname{Top2NTop}(\mathbb{R}^{\mathbf{1}})$.

(55) A is open if and only if for every real number x such that $x \in A$ there exists r such that $r > 0$ and $]x - r, x + r[\subseteq A$.

(56) $\{]a, b[$, where a, b are real numbers : $a < b\}$ is a basis of $\mathbb{R}^{\mathbf{1}}$.

(57) $\{]a, b[$, where a, b are real numbers : $a < b\}$ is a basis of FMT-$\mathbb{R}^1$.

Proof: Set $B = \{]a, b[$, where a, b are real numbers : $a < b\}$. $B \subseteq 2^{\alpha}$, where α is the carrier of FMT-$\mathbb{R}^1$. $B \subseteq$ the open set family of FMT-$\mathbb{R}^1$. $\square$

(58) If $r > 0$, then $]x - r, x + r[$ is a neighbourhood of x. The theorem is a consequence of (57).

(59) Let us consider an object x. Then x is a point of FMT-$\mathbb{R}^1$ if and only if x is a point of the metric space of real numbers.

(60) If $x = y$, then $\operatorname{Ball}(y, r) =]x - r, x + r[$.

(61) If $x = y$ and $r > 0$, then $\operatorname{Ball}(y, r)$ is a neighbourhood of x. The theorem is a consequence of (58).

(62) If $x = z$, then Balls z is a family of subsets of FMT-$\mathbb{R}^1$.

(63) Let us consider a family S of subsets of FMT-$\mathbb{R}^1$. If $x = z$ and $S =$ Balls z, then $[S] = U_F(x)$. The theorem is a consequence of (61), (14), and (55).

The functor gen-NS-$\mathbb{R}^1$ yielding a function from the carrier of FMT-$\mathbb{R}^1$ into $2^{2^{(\text{the carrier of FMT-}\mathbb{R}^1)}}$ is defined by

(Def. 20) for every real number r, there exists a point x of (the metric space of real numbers)$_{\text{top}}$ such that $x = r$ and $it(r) = \text{Balls}\, x$.

The functor gen-$\mathbb{R}^1$ yielding a non empty, strict formal topological space is defined by the term

(Def. 21) $\langle$the carrier of FMT-$\mathbb{R}^1$, gen-NS-$\mathbb{R}^1\rangle$.

Now we state the propositions:

(64) The carrier of gen-$\mathbb{R}^1 = \mathbb{R}$.

(65) Let us consider an element x of gen-$\mathbb{R}^1$. Then there exists a point y of (the metric space of real numbers)$_{\text{top}}$ such that

  (i) $x = y$, and

  (ii) $U_F(x) = \text{Balls}\, y$.

(66) dom[gen-$\mathbb{R}^1$] $= \mathbb{R}$.

(67) gen-filter gen-$\mathbb{R}^1$ = FMT-$\mathbb{R}^1$. The theorem is a consequence of (64), (65), and (58).

6. Transporting Urysohn's Lemma ([4, URYSOHN3:20]) from an Open-Set Topological Space to the Associated Neighborhood Topological Space

Now we state the proposition:

(68) **Main result** URYSOHN'S LEMMA IN A NEIGHBORHOOD TOPOLOGICAL SPACE:
Let us consider a non empty, normal neighborhood topological space N, and closed subsets A, B of N. Suppose A misses B. Then there exists a function F from N into FMT-$\mathbb{R}^1$ such that

  (i) F is continuous, and

  (ii) for every point x of N, $0 \leqslant F(x) \leqslant 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

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