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Unification of Graphs and Relations in Mizar

Sebastian Koch
Johannes Gutenberg University
Mainz, Germany[1]

Summary. A (di)graph without parallel edges can simply be represented by a binary relation of the vertices and on the other hand, any binary relation can be expressed as such a graph. In this article, this correspondence is formalized in the Mizar system [2], based on the formalization of graphs in [6] and relations in [11], [12]. Notably, a new definition of `createGraph` will be given, taking only a non empty set V and a binary relation $E \subseteq V \times V$ to create a (di)graph without parallel edges, which will provide to be very useful in future articles.

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0. Introduction

Digraphs without multiple edges can be represented by binary relations (cf. [4]) and this is in fact the way they are usually defined in textbooks which are primarily concerned about graphs without multiple edges (cf. [10], [3], [8]). While a mathematician can switch between these representations without problems, due to its pedantic nature the Mizar system [2] needs a formalization of this change of viewpoint, which is provided by this article. In the Mizar Mathematical Library [1] this problem hasn't been adressed yet, although the undirected analogon can be found as an alternative definition for simple graphs in [9] (which

[1]The author is enrolled in the Johannes Gutenberg University in Mayence, Germany, mailto: `skoch02@students.uni-mainz.de`

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isn't used anywhere else) and the friendship theorem was formalized in [7] using only relations.

In the first section the dominance and adjacency relation of a graph G are rigorously introduced. G isn't required to be without parallel edges for this, therefore the relations of G and the graph given by removing parallel edges (directed parallel for the dominance) as defined in [5] are the same.

The second section introduces the new functor definition for `createGraph`, taking a non empty set V and a relation $E \subseteq V \times V$ and returning a graph representing this relation. It is shown that the graph created this way from a dominance relation of a graph G without directed parallel edges is directed isomorphic to G itself.

Since undirected graphs are sometimes viewed as symmetric digraphs (cf. [3], [4], [8], the last section introduces a mode getting a graph without parallel edges of any kind by simply removing them from the functor result of the previous section. Similar to before, it is shown that the graph created this way from an adjacency relation of a graph G without parallel edges is isomorphic to G itself.

1. The Adjacency Relation

From now on G denotes a graph.

Let us consider G. The functor VertDomRel(G) yielding a binary relation on the vertices of G is defined by the term

(Def. 1) (the source of G **qua** binary relation)$^{\smile} \cdot$ (the target of G).

Let us consider objects v, w. Now we state the propositions:

(1) $\langle v, w\rangle \in$ VertDomRel(G) if and only if there exists an object e such that e joins v to w in G.

(2) $\langle v, w\rangle \in$ (VertDomRel(G))$^{\smile}$ if and only if there exists an object e such that e joins w to v in G. The theorem is a consequence of (1).

(3) G is loopless if and only if VertDomRel(G) is irreflexive.

Let G be a loopless graph. One can verify that VertDomRel(G) is irreflexive.

Let G be a non loopless graph. One can verify that VertDomRel(G) is non irreflexive.

Let G be a non-multi graph. One can verify that VertDomRel(G) is antisymmetric.

Let G be a simple graph. One can check that VertDomRel(G) is asymmetric.

Now we state the proposition:

(4) Let us consider a graph G. Suppose there exist objects e_1, e_2, x, y such that e_1 joins x to y in G and e_2 joins y to x in G. Then VertDomRel(G) is not asymmetric.

PROOF: Set $R = \mathrm{VertDomRel}(G)$. There exist objects x, y such that x, $y \in \mathrm{field}\, R$ and $\langle x, y\rangle$, $\langle y, x\rangle \in R$. □

Let G be a non non-multi, non-directed-multi graph.

Note that VertDomRel(G) is non asymmetric.

Now we state the propositions:

(5) Let us consider a loopless graph G. Suppose field VertDomRel(G) = the vertices of G. Then every component of G is not trivial. The theorem is a consequence of (1).

(6) Let us consider a graph G. Suppose every component of G is not trivial. Then field VertDomRel(G) = the vertices of G. The theorem is a consequence of (1).

(7) Let us consider a non trivial, connected graph G. Then field VertDomRel (G) = the vertices of G. The theorem is a consequence of (6).

Let G be a complete graph. One can verify that VertDomRel(G) is connected.

(8) G is edgeless if and only if VertDomRel(G) is empty. The theorem is a consequence of (1).

Let G be an edgeless graph. Let us observe that VertDomRel(G) is empty.

Let G be a non edgeless graph. One can verify that VertDomRel(G) is non empty.

Now we state the proposition:

(9) G is loopfull if and only if VertDomRel(G) is total and reflexive.

Let G be a loopfull graph. Note that VertDomRel(G) is reflexive and total.

Let G be a vertex-finite graph. Let us observe that VertDomRel(G) is finite.

(10) $\overline{\overline{\mathrm{VertDomRel}(G)}} = \overline{\overline{\mathrm{Classes\,DEdgeParEqRel}(G)}}$.

PROOF: Set $R = \mathrm{VertDomRel}(G)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an object e such that e joins $(\$_1)_{\mathbf{1}}$ to $(\$_1)_{\mathbf{2}}$ in G and $\$_2 = [e]_{\mathrm{DEdgeParEqRel}(G)}$. For every objects x, y_1, y_2 such that $x \in R$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in R$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\mathrm{dom}\, f = R$ and for every object x such that $x \in R$ holds $\mathcal{P}[x, f(x)]$. For every objects x_1, x_2 such that x_1, $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. □

(11) $\overline{\overline{\mathrm{VertDomRel}(G)}} \subseteq G$.size(). The theorem is a consequence of (10).

(12) Let us consider a non-directed-multi graph G. Then G.size() = $\overline{\overline{\mathrm{VertDomRel}(G)}}$. The theorem is a consequence of (10).

Let us consider a vertex v of G. Now we state the propositions:

(13) $(\mathrm{VertDomRel}(G))^{\circ} v = v$.outNeighbors(). The theorem is a consequence of (1).

(14) $\mathrm{Coim}(\mathrm{VertDomRel}(G), v) = v.\mathrm{inNeighbors}()$. The theorem is a consequence of (1).

(15) Let us consider a subgraph H of G. Then $\mathrm{VertDomRel}(H) \subseteq \mathrm{VertDomRel}(G)$. The theorem is a consequence of (1).

(16) Let us consider a subgraph H of G with directed-parallel edges removed. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem is a consequence of (15) and (1).

(17) Let us consider a subgraph H of G with loops removed. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\mathrm{id}_\alpha)$, where α is the vertices of G. The theorem is a consequence of (1) and (15).

(18) Let us consider a directed-simple graph H of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\mathrm{id}_\alpha)$, where α is the vertices of G. The theorem is a consequence of (17) and (16).

(19) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then $\mathrm{VertDomRel}(G_1) = \mathrm{VertDomRel}(G_2)$. The theorem is a consequence of (1).

(20) Let us consider a graph H given by reversing directions of the edges of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G))^{\smile}$. The theorem is a consequence of (1).

(21) Let us consider a non empty subset V of the vertices of G, and a subgraph H of G induced by V. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G) \cap (V \times V)$. The theorem is a consequence of (1) and (15).

(22) Let us consider a set V, and a subgraph H of G with vertices V removed. Suppose $V \subset$ the vertices of G. Then $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (V \times (\text{the vertices of } G) \cup (\text{the vertices of } G) \times V)$. The theorem is a consequence of (15) and (1).

Let us consider a non trivial graph G, a vertex v of G, and a subgraph H of G with vertex v removed. Now we state the propositions:

(23) $\mathrm{VertDomRel}(H) = (\mathrm{VertDomRel}(G)) \setminus (\{v\} \times (\text{the vertices of } G) \cup (\text{the vertices of } G) \times \{v\})$. The theorem is a consequence of (22).

(24) If v is isolated, then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$.
PROOF: Set $V_1 = \{v\} \times (\text{the vertices of } G)$. Set $V_2 = (\text{the vertices of } G) \times \{v\}$. $(V_1 \cup V_2) \cap \mathrm{VertDomRel}(G) = \emptyset$. $\square$

(25) Let us consider a set V, and a supergraph H of G extended by the vertices from V. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem is a consequence of (15) and (1).

(26) Let us consider objects v, e, w, and a supergraph H of G extended by e between vertices v and w. Suppose there exists an object e_0 such that e_0 joins v to w in G. Then $\mathrm{VertDomRel}(H) = \mathrm{VertDomRel}(G)$. The theorem

is a consequence of (15), (1), and (19).

(27) Let us consider vertices v, w of G, an object e, and a supergraph H of G extended by e between vertices v and w. Suppose $e \notin$ the edges of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (1) and (15).

(28) Let us consider a vertex v of G, objects e, w, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $w \notin$ the vertices of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (27) and (25).

(29) Let us consider objects v, e, a vertex w of G, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $v \notin$ the vertices of G. Then VertDomRel(H) = VertDomRel(G) $\cup$ $\{\langle v, w\rangle\}$. The theorem is a consequence of (27) and (25).

(30) Let us consider a subset V of the vertices of G, and a graph H by adding a loop to each vertex of G in V. Then VertDomRel(H) = VertDomRel(G) $\cup$ id_V. The theorem is a consequence of (1) and (15).

(31) Let us consider a directed graph complement H of G with loops. Then VertDomRel(H) = ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (VertDomRel (G)). The theorem is a consequence of (1).

Let us consider G. The functor VertAdjSymRel(G) yielding a binary relation on the vertices of G is defined by the term

(Def. 2) VertDomRel(G) $\cup$ (VertDomRel(G))$^{\smile}$.

Now we state the propositions:

(32) Let us consider objects v, w. Then $\langle v, w\rangle \in$ VertAdjSymRel(G) if and only if there exists an object e such that e joins v and w in G. The theorem is a consequence of (1) and (2).

(33) Let us consider vertices v, w of G. Then $\langle v, w\rangle \in$ VertAdjSymRel(G) if and only if v and w are adjacent. The theorem is a consequence of (32).

(34) VertDomRel(G) $\subseteq$ VertAdjSymRel(G).

(35) VertAdjSymRel(G) = (the source of G **qua** binary relation)$^{\smile}\cdot$(the target of G) $\cup$ (the target of G **qua** binary relation)$^{\smile}$ $\cdot$ (the source of G).

Let us consider G. One can check that VertAdjSymRel(G) is symmetric.

Now we state the proposition:

(36) G is loopless if and only if VertAdjSymRel(G) is irreflexive.

Let G be a loopless graph. One can verify that VertAdjSymRel(G) is irreflexive.

Let G be a non loopless graph. One can check that VertAdjSymRel(G) is non irreflexive.

Now we state the propositions:

(37) Let us consider a loopless graph G. Suppose VertAdjSymRel(G) is total. Then every component of G is not trivial. The theorem is a consequence of (5).

(38) Let us consider a graph G. Suppose every component of G is not trivial. Then VertAdjSymRel(G) is total. The theorem is a consequence of (6).

Let G be a non trivial, connected graph. Note that VertAdjSymRel(G) is total.

Let G be a complete graph. Let us note that VertAdjSymRel(G) is connected.

Now we state the proposition:

(39) G is edgeless if and only if VertAdjSymRel(G) is empty.

Let G be an edgeless graph. One can check that VertAdjSymRel(G) is empty.

Let G be a non edgeless graph. Note that VertAdjSymRel(G) is non empty.

(40) G is loopfull if and only if VertAdjSymRel(G) is total and reflexive.

Let G be a loopfull graph. Let us observe that VertAdjSymRel(G) is reflexive and total.

Let G be a vertex-finite graph. Note that VertAdjSymRel(G) is finite.

Now we state the propositions:

(41) $\overline{\overline{\text{Classes DEdgeParEqRel}(G)}} \subseteq \overline{\overline{\text{VertAdjSymRel}(G)}}$. The theorem is a consequence of (34) and (10).

(42) $\overline{\overline{\text{Classes EdgeParEqRel}(G)}} \subseteq \overline{\overline{\text{VertAdjSymRel}(G)}}$.

PROOF: Set $R =$ VertAdjSymRel(G). Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an object e such that e joins $(\$_1)_{\mathbf{1}}$ and $(\$_1)_{\mathbf{2}}$ in G and $\$_2 = [e]_{\text{EdgeParEqRel}(G)}$. For every objects x, y_1, y_2 such that $x \in R$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in R$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\text{dom} f = R$ and for every object x such that $x \in R$ holds $\mathcal{P}[x, f(x)]$. □

(43) Let us consider a non-directed-multi graph G. Then $\underline{G.\text{size}()} \subseteq \overline{\overline{\text{VertAdjSymRel}(G)}}$. The theorem is a consequence of (10), (12), and (41).

(44) Let us consider a vertex v of G. Then $(\text{VertAdjSymRel}(G))^\circ v = v.\text{allNeighbors}()$. The theorem is a consequence of (32).

(45) Let us consider a subgraph H of G. Then VertAdjSymRel(H) $\subseteq$ VertAdjSymRel(G). The theorem is a consequence of (15).

(46) Let us consider a subgraph H of G with parallel edges removed. Then VertAdjSymRel(H) $=$ VertAdjSymRel(G). The theorem is a consequence of (45) and (32).

(47) Let us consider a subgraph H of G with loops removed.

Then VertAdjSymRel(H) = (VertAdjSymRel(G))$\setminus$(id_α), where α is the vertices of G. The theorem is a consequence of (17).

(48) Let us consider a simple graph H of G. Then VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ (id_α), where α is the vertices of G. The theorem is a consequence of (47) and (46).

(49) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then VertAdjSymRel (G_1) = VertAdjSymRel(G_2). The theorem is a consequence of (19).

(50) Let us consider a set E, and a graph H given by reversing directions of the edges E of G. Then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (32).

(51) Let us consider a non empty subset V of the vertices of G, and a subgraph H of G induced by V. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cap$ ($V \times V$). The theorem is a consequence of (21).

(52) Let us consider a set V, and a subgraph H of G with vertices V removed. Suppose $V \subset$ the vertices of G. Then VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ ($V \times$ (the vertices of G) $\cup$ (the vertices of G) $\times V$). The theorem is a consequence of (22).

Let us consider a non trivial graph G, a vertex v of G, and a subgraph H of G with vertex v removed. Now we state the propositions:

(53) VertAdjSymRel(H) = (VertAdjSymRel(G)) $\setminus$ ($\{v\} \times$ (the vertices of G) $\cup$ (the vertices of G) $\times \{v\}$). The theorem is a consequence of (52).

(54) If v is isolated, then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (24).

(55) Let us consider a set V, and a supergraph H of G extended by the vertices from V. Then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (25).

Let us consider vertices v, w of G, an object e, and a supergraph H of G extended by e between vertices v and w. Now we state the propositions:

(56) If v and w are adjacent, then VertAdjSymRel(H) = VertAdjSymRel(G). The theorem is a consequence of (26), (1), (27), and (49).

(57) Suppose $e \notin$ the edges of G. Then VertAdjSymRel(H) = VertAdjSymRel (G) $\cup$ $\{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (27).

(58) Let us consider a vertex v of G, objects e, w, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $w \notin$ the vertices of G. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cup$ $\{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (57) and (55).

(59) Let us consider objects v, e, a vertex w of G, and a supergraph H of G extended by v, w and e between them. Suppose $e \notin$ the edges of G and $v \notin$

the vertices of G. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cup \{\langle v, w\rangle, \langle w, v\rangle\}$. The theorem is a consequence of (57) and (55).

(60) Let us consider an object v, a subset V of the vertices of G, and a supergraph H of G extended by vertex v and edges between v and V of G. Suppose $v \notin$ the vertices of G. Then VertAdjSymRel(H) = (VertAdjSymRel (G) $\cup \{v\} \times V) \cup V \times \{v\}$. The theorem is a consequence of (32) and (45).

(61) Let us consider a subset V of the vertices of G, and a graph H by adding a loop to each vertex of G in V. Then VertAdjSymRel(H) = VertAdjSymRel(G) $\cup\, \mathrm{id}_V$. The theorem is a consequence of (30).

(62) Let us consider an undirected graph complement H of G with loops. Then VertAdjSymRel(H) = ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (VertAdjSymRel(G)). The theorem is a consequence of (32).

2. Create non-Directed-Multi Graphs from Relations

In the sequel V denotes a non empty set and E denotes a binary relation on V.

Let us consider V and E. The functor createGraph(V, E) yielding a graph is defined by the term

(Def. 3) createGraph$(V, E, \pi_1(V \boxtimes V){\restriction}E, \pi_2(V \boxtimes V){\restriction}E)$.

Let us note that the edges of createGraph(V, E) is relation-like.

Now we state the propositions:

(63) Let us consider objects v, w. Then $\langle v, w\rangle \in E$ if and only if $\langle v, w\rangle$ joins v to w in createGraph(V, E).

(64) Let us consider objects e, v, w. Suppose e joins v to w in createGraph(V, E). Then $e = \langle v, w\rangle$. The theorem is a consequence of (63).

(65) VertDomRel(createGraph(V, E)) $= E$. The theorem is a consequence of (1) and (63).

Let us consider V and E. One can verify that createGraph(V, E) is plain and non-directed-multi.

Now we state the proposition:

(66) V is trivial if and only if createGraph(V, E) is trivial.

Let V be a trivial, non empty set and E be a binary relation on V. One can check that createGraph(V, E) is trivial.

Let V be a non trivial set. Let us observe that createGraph(V, E) is non trivial.

Now we state the proposition:

(67) E is irreflexive if and only if createGraph(V, E) is loopless. The theorem is a consequence of (65).

Let us consider V. Let E be an irreflexive binary relation on V. Let us note that createGraph(V, E) is loopless.

Let E be a non irreflexive binary relation on V. Observe that createGraph(V, E) is non loopless.

Now we state the proposition:

(68) E is antisymmetric if and only if createGraph(V, E) is non-multi. The theorem is a consequence of (64) and (65).

Let us consider V. Let E be an antisymmetric binary relation on V. One can check that createGraph(V, E) is non-multi.

Let V be a non trivial set and E be a non antisymmetric binary relation on V. Note that createGraph(V, E) is non non-multi.

Let us consider V. Let E be an asymmetric binary relation on V. One can verify that createGraph(V, E) is simple.

Now we state the proposition:

(69) If createGraph(V, E) is complete, then E is connected. The theorem is a consequence of (65).

Let V be a non trivial set and E be a non connected binary relation on V. Note that createGraph(V, E) is non complete.

Now we state the proposition:

(70) E is empty if and only if createGraph(V, E) is edgeless. The theorem is a consequence of (65).

Let us consider V. Let E be an empty binary relation on V. One can verify that createGraph(V, E) is edgeless.

Let E be a non empty binary relation on V. Note that createGraph(V, E) is non edgeless.

Now we state the proposition:

(71) E is total and reflexive if and only if createGraph(V, E) is loopfull. The theorem is a consequence of (65).

Let us consider V. Let E be a total, reflexive binary relation on V. Let us note that createGraph(V, E) is loopfull.

Let E be a non total binary relation on V. Observe that createGraph(V, E) is non loopfull.

Let V be a finite, non empty set and E be a binary relation on V. One can check that createGraph(V, E) is finite.

Let us consider V. Let E be a finite binary relation on V. One can check that createGraph(V, E) is edge-finite.

Let us consider a vertex v of createGraph(V, E). Now we state the propositions:

(72) $E^{\circ}v = v$.outNeighbors(). The theorem is a consequence of (63) and (64).

(73) $\mathrm{Coim}(E, v) = v$.inNeighbors(). The theorem is a consequence of (63) and (64).

(74) Let us consider a set X. Then $E \restriction X =$ (createGraph(V, E)).edgesOutOf (X). The theorem is a consequence of (63) and (64).

(75) Let us consider a set Y. Then $Y \upharpoonleft E =$ (createGraph(V, E)).edgesInto(Y). The theorem is a consequence of (63) and (64).

Let us consider sets X, Y. Now we state the propositions:

(76) $(Y \upharpoonleft E) \restriction X =$ (createGraph(V, E)).edgesDBetween(X, Y). The theorem is a consequence of (75) and (74).

(77) $(Y \upharpoonleft E) \restriction X \cup (X \upharpoonleft E) \restriction Y =$ (createGraph(V, E)).edgesBetween(X, Y). The theorem is a consequence of (76).

Let us consider a vertex v of createGraph(V, E). Now we state the propositions:

(78) $E \restriction \{v\} = v$.edgesOut(). The theorem is a consequence of (74).

(79) $\{v\} \upharpoonleft E = v$.edgesIn(). The theorem is a consequence of (75).

(80) Let us consider a set X. Then $E \restriction X \cup X \upharpoonleft E =$ (createGraph(V, E)) .edgesInOut(X). The theorem is a consequence of (74) and (75).

(81) $\mathrm{dom}\, E = \mathrm{rng}$(the source of createGraph(V, E)). The theorem is a consequence of (63) and (64).

(82) $\mathrm{rng}\, E = \mathrm{rng}$(the target of createGraph(V, E)). The theorem is a consequence of (63) and (64).

(83) Let us consider a vertex v of createGraph(V, E). Then v is isolated if and only if $v \notin \mathrm{field}\, E$. The theorem is a consequence of (63) and (64).

(84) E is symmetric if and only if VertAdjSymRel(createGraph(V, E)) $= E$. The theorem is a consequence of (65).

(85) Let us consider a non empty set V_1, a non empty subset V_2 of V_1, a binary relation E_1 on V_1, and a binary relation E_2 on V_2. Suppose $E_2 \subseteq E_1$. Then createGraph(V_2, E_2) is a subgraph of createGraph(V_1, E_1) induced by V_2 and E_2.

Let us consider a non-directed-multi graph G. Now we state the propositions:

(86) There exists a partial graph mapping F from G to createGraph(the vertices of G, VertDomRel(G)) such that

(i) F is directed-isomorphism, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_{\alpha}$, and

(iii) for every object e such that $e \in$ the edges of G holds $(F_{\mathbb{E}})(e) = \langle$(the source of G)(e), (the target of G)$(e)\rangle$,

where α is the vertices of G.

(87) createGraph(the vertices of G, VertDomRel(G)) is G-directed-isomorphic. The theorem is a consequence of (86).

3. Create non-Multi Graphs from Symmetric Relations

In the sequel E denotes a symmetric binary relation on V.

Let us consider V and E.

A graph created from the symmetric relation V on E is a subgraph of createGraph(V, E) with parallel edges removed. From now on G denotes a graph created from the symmetric relation V on E.

Now we state the propositions:

(88) Let us consider objects v, w. Then $\langle v, w\rangle \in E$ if and only if $\langle v, w\rangle$ joins v to w in G or $\langle w, v\rangle$ joins w to v in G. The theorem is a consequence of (63).

(89) Let us consider vertices v, w of G. Then $\langle v, w\rangle \in E$ if and only if v and w are adjacent. The theorem is a consequence of (88) and (63).

Let us consider V and E. Let us observe that every graph created from the symmetric relation V on E is non-multi.

Now we state the proposition:

(90) The edges of $G \subseteq E$.

Let us consider graphs G_1, G_2 created from the symmetric relation V on E. Now we state the propositions:

(91) The vertices of G_1 = the vertices of G_2.

(92) G_2 is G_1-isomorphic.

(93) V is trivial if and only if G is trivial.

Let V be a trivial, non empty set and E be a symmetric binary relation on V. Observe that every graph created from the symmetric relation V on E is trivial.

Let V be a non trivial set. Let us note that every graph created from the symmetric relation V on E is non trivial.

Now we state the proposition:

(94) E is irreflexive if and only if G is loopless.

Let us consider V. Let E be a symmetric, irreflexive binary relation on V. One can verify that every graph created from the symmetric relation V on E is loopless.

Let E be a symmetric, non irreflexive binary relation on V. Observe that every graph created from the symmetric relation V on E is non loopless.

Now we state the proposition:

(95) If G is complete, then E is connected. The theorem is a consequence of (69).

Let V be a non trivial set and E be a symmetric, non connected binary relation on V. Note that every graph created from the symmetric relation V on E is non complete.

Now we state the proposition:

(96) E is empty if and only if G is edgeless.

Let us consider V. Let E be an empty binary relation on V. Let us note that every graph created from the symmetric relation V on E is edgeless.

Let E be a symmetric, non empty binary relation on V. One can check that every graph created from the symmetric relation V on E is non edgeless.

Now we state the proposition:

(97) E is total and reflexive if and only if G is loopfull. The theorem is a consequence of (71).

Let us consider V. Let E be a total, reflexive, symmetric binary relation on V. Observe that every graph created from the symmetric relation V on E is loopfull.

Let E be a symmetric, non total binary relation on V. Note that every graph created from the symmetric relation V on E is non loopfull.

Let V be a finite, non empty set and E be a symmetric binary relation on V. One can verify that every graph created from the symmetric relation V on E is finite.

Now we state the propositions:

(98) Let us consider a vertex v of G. Then $E^{\circ}v = v.\mathrm{allNeighbors}()$. The theorem is a consequence of (72) and (73).

(99) Let us consider a set X. Then $G.\mathrm{edgesInOut}(X) \subseteq E{\upharpoonright}X \cup X{\restriction}E$. The theorem is a consequence of (80).

(100) Let us consider sets X, Y. Then $G.\mathrm{edgesBetween}(X, Y) \subseteq (Y{\restriction}E){\upharpoonright}X \cup (X{\restriction}E){\upharpoonright}Y$. The theorem is a consequence of (77).

Let us consider a vertex v of G. Now we state the propositions:

(101) $v.\mathrm{edgesOut}() \subseteq E{\upharpoonright}\{v\}$. The theorem is a consequence of (78).

(102) $v.\mathrm{edgesIn}() \subseteq \{v\}{\restriction}E$. The theorem is a consequence of (79).

(103) v is isolated if and only if $v \notin \mathrm{field}\, E$. The theorem is a consequence of (83).

(104) Let us consider a graph G created from the symmetric relation V on E. Then $\mathrm{VertAdjSymRel}(G) = E$. The theorem is a consequence of (33) and (89).

(105) Let us consider a non empty set V_1, a non empty subset V_2 of V_1, a symmetric binary relation E_1 on V_1, a symmetric binary relation E_2 on V_2, a graph G_1 created from the symmetric relation V_1 on E_1, and a graph G_2 created from the symmetric relation V_2 on E_2. Suppose $E_2 \subseteq E_1$. Then there exists a partial graph mapping F from G_2 to G_1 such that

(i) F is weak subgraph embedding, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_{V_2}$, and

(iii) for every objects v, w such that $\langle v, w\rangle \in$ the edges of G_2 holds $(F_{\mathbb{E}})(\langle v, w\rangle) = \langle v, w\rangle$ or $(F_{\mathbb{E}})(\langle v, w\rangle) = \langle w, v\rangle$.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist objects v, w such that $\$_1 = \langle v, w\rangle$ and $\$_2 \in$ the edges of G_1 and ($\$_2 = \langle v, w\rangle$ or $\$_2 = \langle w, v\rangle$). For every objects x, y_1, y_2 such that $x \in$ the edges of G_2 and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that $x \in$ the edges of G_2 there exists an object y such that $\mathcal{P}[x, y]$. Consider g being a function such that $\mathrm{dom}\, g =$ the edges of G_2 and for every object x such that $x \in$ the edges of G_2 holds $\mathcal{P}[x, g(x)]$. For every objects x_1, x_2 such that x_1, $x_2 \in \mathrm{dom}\, g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$. Consider v_0, w_0 being objects such that $\langle v, w\rangle = \langle v_0, w_0\rangle$ and $g(\langle v, w\rangle) \in$ the edges of G_1 and $g(\langle v, w\rangle) = \langle v_0, w_0\rangle$ or $g(\langle v, w\rangle) = \langle w_0, v_0\rangle$. □

(106) Let us consider a non-multi graph G_1, and a graph G_2 created from the symmetric relation the vertices of G_1 on $\mathrm{VertAdjSymRel}(G_1)$. Then there exists a partial graph mapping F from G_1 to G_2 such that

(i) F is isomorphism, and

(ii) $F_{\mathbb{V}} = \mathrm{id}_{\alpha}$, and

(iii) for every object e such that $e \in$ the edges of G_1 holds $(F_{\mathbb{E}})(e) = \langle$(the source of $G_1)(e)$, (the target of $G_1)(e)\rangle$ or $(F_{\mathbb{E}})(e) = \langle$(the target of $G_1)(e)$, (the source of $G_1)(e)\rangle$,

where α is the vertices of G_1.

Proof: Set $E_0 = \mathrm{VertAdjSymRel}(G)$. Set $G_0 = \mathrm{createGraph}$(the vertices of $G, E_0)$. Consider E' being a representative selection of the parallel edges of G_0 such that G' is a subgraph of G_0 induced by the vertices of G_0 and E'. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 \in E'$ and ($\$_2 = \langle$(the source of $G)(\$_1)$, (the target of $G)(\$_1)\rangle$ or $\$_2 = \langle$(the target of $G)(\$_1)$, (the source of $G)(\$_1)\rangle$). For every objects x, y_1, y_2 such that $x \in$ the edges of G and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. For every object x such that

$x \in$ the edges of G there exists an object y such that $\mathcal{P}[x, y]$. Consider g being a function such that $\operatorname{dom} g =$ the edges of G and for every object x such that $x \in$ the edges of G holds $\mathcal{P}[x, g(x)]$. $\square$

(107) Let us consider a non-multi graph G_1. Then every graph created from the symmetric relation the vertices of G_1 on VertAdjSymRel(G_1) is G_1-isomorphic. The theorem is a consequence of (106).

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