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About Graph Unions and Intersections

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Summary. In this article the union and intersection of a set of graphs are formalized in the Mizar system [5], based on the formalization of graphs in [7].

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0. Introduction

The union and intersection of two graphs are usually defined in any general graph theory textbook, although there are small differences between the authors from time to time. For example, Wilson [10] only allows two vertex- and edge-disjoint graphs to be united; his graph union is usually known as the disjoint union [2], [8] or sum [8] of two graphs, which will be formalized in in detail in another article. Bondy and Murty [2] as well as Diestel [4] allow unions of two arbitary simple graphs, but labelled the vertices in the graphical representation to avoid confusion. In both books it was silently assumed that edges between the same vertices in both graphs are the same, thereby securing the union to be a simple graph again. Wagner [9], while generalizing to the union and intersection of a family of graphs, explicitly states that condition and previously adds the condition, that on the other side identical edges in the graph family must have the same incident vertices. Naturally, in this paper union and intersection are generalized to families of multidigraphs, i.e. the graphs of [7]. Union and

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intersection are defined as modes rather than functions in accordance with the style of the early `GLIB` articles and to leave this formalization extendable by graph decorators.

To denote the graph family, a `Graph-yielding Function` from [7] could have been used. But since sets of graphs would be needed sooner or later in the Mizar Mathematical Library [1] (e.g. to count all spanning trees of a graph), the set attribute `Graph-membered` is rigorously introduced in the first section.

In the second section, the first condition of Wagner is formalized. It simply means that for two graphs G and H from the family, their respective source and target function tolerate each other ($S(G) \approx S(H)$ and $T(G) \approx T(H)$, cf. [3]). As this property is indispensable for unions (or else in a union an edge could point to different vertices), the set attribute was named `\/-tolerating`. The graph union U for a $\cup$-tolerating set S is given by

$$U = (\bigcup_{G \in S} V(G), \bigcup_{G \in S} E(G), \bigcup_{G \in S} S(G), \bigcup_{G \in S} T(G)).$$

While Wagner's second condition is useful to ensure the resulting graph union will be non-multi, it is not formalized in this article.

Since graphs without vertices are not allowed by the used definition [7], the difference between $\cup$-tolerating and `/\-tolerating` is the additional condition that $\bigcap_{G \in S} V(G)$ is non empty. Then the graph intersection I for a $\cap$-tolerating set S is given by

$$I = (\bigcap_{G \in S} V(G), \bigcap_{G \in S} E(G), \bigcap_{G \in S} S(G), \bigcap_{G \in S} T(G)).$$

To avoid confusion with intersection graphs of any kind, the mode was named `GraphMeet`.

With this formalization the union of a graph with (any kind of) its complement will be complete and the intersection will be edgeless, just as intended by [6].

1. Sets of Graphs

Let X be a set. We say that X is graph-membered if and only if

(Def. 1) for every object x such that $x \in X$ holds x is a graph.

Observe that every set which is empty is also graph-membered.

Let F be a graph-yielding function. One can verify that $\operatorname{rng} F$ is graph-membered.

Let G_1 be a graph. Let us note that $\{G_1\}$ is graph-membered.

Let G_2 be a graph. Let us observe that $\{G_1, G_2\}$ is graph-membered and there exists a set which is empty and graph-membered and there exists a set which is trivial, finite, non empty, and graph-membered.

Let X be a graph-membered set. One can check that every subset of X is graph-membered.

Let Y be a set. Let us note that $X \cap Y$ is graph-membered and $X \setminus Y$ is graph-membered.

Let X, Y be graph-membered sets. Let us note that $X \cup Y$ is graph-membered and $X \dot{-} Y$ is graph-membered.

Let us consider a set X. Now we state the propositions:

(1) If for every object Y such that $Y \in X$ holds Y is a graph-membered set, then $\bigcup X$ is graph-membered.

(2) If there exists a graph-membered set Y such that $Y \in X$, then $\bigcap X$ is graph-membered.

Let X be a non empty, graph-membered set. Observe that every element of X is function-like and relation-like and every element of X is $\mathbb{N}$-defined and finite and every element of X is graph-like.

Let S be a graph-membered set. We say that S is plain if and only if

(Def. 2) for every graph G such that $G \in S$ holds G is plain.

We say that S is loopless if and only if

(Def. 3) for every graph G such that $G \in S$ holds G is loopless.

We say that S is non-multi if and only if

(Def. 4) for every graph G such that $G \in S$ holds G is non-multi.

We say that S is non-directed-multi if and only if

(Def. 5) for every graph G such that $G \in S$ holds G is non-directed-multi.

We say that S is simple if and only if

(Def. 6) for every graph G such that $G \in S$ holds G is simple.

We say that S is directed-simple if and only if

(Def. 7) for every graph G such that $G \in S$ holds G is directed-simple.

We say that S is acyclic if and only if

(Def. 8) for every graph G such that $G \in S$ holds G is acyclic.

We say that S is connected if and only if

(Def. 9) for every graph G such that $G \in S$ holds G is connected.

We say that S is tree-like if and only if

(Def. 10) for every graph G such that $G \in S$ holds G is tree-like.

We say that S is chordal if and only if

(Def. 11) for every graph G such that $G \in S$ holds G is chordal.

We say that S is edgeless if and only if

(Def. 12) for every graph G such that $G \in S$ holds G is edgeless.

We say that S is loopfull if and only if

(Def. 13) for every graph G such that $G \in S$ holds G is loopfull.

Let us observe that every graph-membered set which is empty is also plain, loopless, non-multi, non-directed-multi, simple, directed-simple, acyclic, connected, tree-like, chordal, edgeless, and loopfull and every graph-membered set which is non-multi is also non-directed-multi and every graph-membered set which is loopless and non-multi is also simple and every graph-membered set which is loopless and non-directed-multi is also directed-simple.

Every graph-membered set which is simple is also loopless and non-multi and every graph-membered set which is directed-simple is also loopless and non-directed-multi and every graph-membered set which is acyclic is also simple and every graph-membered set which is acyclic and connected is also tree-like and every graph-membered set which is tree-like is also acyclic and connected.

Let G_1 be a plain graph. Let us observe that $\{G_1\}$ is plain. Let G_2 be a plain graph. One can check that $\{G_1, G_2\}$ is plain.

Let G_1 be a loopless graph. One can verify that $\{G_1\}$ is loopless. Let G_2 be a loopless graph. Note that $\{G_1, G_2\}$ is loopless.

Let G_1 be a non-multi graph. One can check that $\{G_1\}$ is non-multi. Let G_2 be a non-multi graph. Let us note that $\{G_1, G_2\}$ is non-multi.

Let G_1 be a non-directed-multi graph. Note that $\{G_1\}$ is non-directed-multi. Let G_2 be a non-directed-multi graph. Observe that $\{G_1, G_2\}$ is non-directed-multi.

Let G_1 be a simple graph. Let us note that $\{G_1\}$ is simple. Let G_2 be a simple graph. One can verify that $\{G_1, G_2\}$ is simple.

Let G_1 be a directed-simple graph. Let us observe that $\{G_1\}$ is directed-simple. Let G_2 be a directed-simple graph. Note that $\{G_1, G_2\}$ is directed-simple.

Let G_1 be an acyclic graph. One can check that $\{G_1\}$ is acyclic. Let G_2 be an acyclic graph. Let us note that $\{G_1, G_2\}$ is acyclic.

Let G_1 be a connected graph. Note that $\{G_1\}$ is connected. Let G_2 be a connected graph. Observe that $\{G_1, G_2\}$ is connected.

Let G_1 be a tree-like graph. Let us note that $\{G_1\}$ is tree-like. Let G_2 be a tree-like graph. One can verify that $\{G_1, G_2\}$ is tree-like.

Let G_1 be a chordal graph. Let us observe that $\{G_1\}$ is chordal. Let G_2 be a chordal graph. One can check that $\{G_1, G_2\}$ is chordal.

Let G_1 be an edgeless graph. One can verify that $\{G_1\}$ is edgeless. Let G_2 be an edgeless graph. Note that $\{G_1, G_2\}$ is edgeless.

Let G_1 be a loopfull graph. One can check that $\{G_1\}$ is loopfull. Let G_2 be a loopfull graph. Let us note that $\{G_1, G_2\}$ is loopfull.

Let F be a plain, graph-yielding function. Observe that $\operatorname{rng} F$ is plain.

Let F be a loopless, graph-yielding function. One can verify that $\operatorname{rng} F$ is loopless.

Let F be a non-multi, graph-yielding function. Note that $\operatorname{rng} F$ is non-multi.

Let F be a non-directed-multi, graph-yielding function. Observe that $\operatorname{rng} F$ is non-directed-multi.

Let F be a simple, graph-yielding function. One can verify that $\operatorname{rng} F$ is simple.

Let F be a directed-simple, graph-yielding function. Observe that $\operatorname{rng} F$ is directed-simple.

Let F be an acyclic, graph-yielding function. Note that $\operatorname{rng} F$ is acyclic.

Let F be a connected, graph-yielding function. Observe that $\operatorname{rng} F$ is connected.

Let F be a tree-like, graph-yielding function. One can verify that $\operatorname{rng} F$ is tree-like.

Let F be a chordal, graph-yielding function. Observe that $\operatorname{rng} F$ is chordal.

Let F be an edgeless, graph-yielding function. One can verify that $\operatorname{rng} F$ is edgeless.

Let F be a loopfull, graph-yielding function. Note that $\operatorname{rng} F$ is loopfull.

Let X be a plain, graph-membered set. Observe that every subset of X is plain.

Let X be a loopless, graph-membered set. Note that every subset of X is loopless.

Let X be a non-multi, graph-membered set. One can verify that every subset of X is non-multi.

Let X be a non-directed-multi, graph-membered set. Observe that every subset of X is non-directed-multi.

Let X be a simple, graph-membered set. Note that every subset of X is simple.

Let X be a directed-simple, graph-membered set. One can check that every subset of X is directed-simple.

Let X be an acyclic, graph-membered set. One can verify that every subset of X is acyclic.

Let X be a connected, graph-membered set. Observe that every subset of X is connected.

Let X be a tree-like, graph-membered set. Note that every subset of X is tree-like.

Let X be a chordal, graph-membered set. One can check that every subset of X is chordal.

Let X be an edgeless, graph-membered set. Let us observe that every subset of X is edgeless.

Let X be a loopfull, graph-membered set. Let us note that every subset of X is loopfull.

Let X be a plain, graph-membered set and Y be a set. Note that $X \cap Y$ is plain and $X \setminus Y$ is plain.

Let X, Y be plain, graph-membered sets. Observe that $X \cup Y$ is plain and $X \dot{-} Y$ is plain.

Let X be a loopless, graph-membered set and Y be a set. Note that $X \cap Y$ is loopless and $X \setminus Y$ is loopless.

Let X, Y be loopless, graph-membered sets. Observe that $X \cup Y$ is loopless and $X \dot{-} Y$ is loopless.

Let X be a non-multi, graph-membered set and Y be a set. Note that $X \cap Y$ is non-multi and $X \setminus Y$ is non-multi.

Let X, Y be non-multi, graph-membered sets. Observe that $X \cup Y$ is non-multi and $X \dot{-} Y$ is non-multi.

Let X be a non-directed-multi, graph-membered set and Y be a set. Note that $X \cap Y$ is non-directed-multi and $X \setminus Y$ is non-directed-multi.

Let X, Y be non-directed-multi, graph-membered sets. Observe that $X \cup Y$ is non-directed-multi and $X \dot{-} Y$ is non-directed-multi.

Let X be a simple, graph-membered set and Y be a set. Note that $X \cap Y$ is simple and $X \setminus Y$ is simple.

Let X, Y be simple, graph-membered sets. Observe that $X \cup Y$ is simple and $X \dot{-} Y$ is simple.

Let X be a directed-simple, graph-membered set and Y be a set. Note that $X \cap Y$ is directed-simple and $X \setminus Y$ is directed-simple.

Let X, Y be directed-simple, graph-membered sets. Observe that $X \cup Y$ is directed-simple and $X \dot{-} Y$ is directed-simple.

Let X be an acyclic, graph-membered set and Y be a set. Note that $X \cap Y$ is acyclic and $X \setminus Y$ is acyclic.

Let X, Y be acyclic, graph-membered sets. Observe that $X \cup Y$ is acyclic and $X \dot{-} Y$ is acyclic.

Let X be a connected, graph-membered set and Y be a set. Note that $X \cap Y$ is connected and $X \setminus Y$ is connected.

Let X, Y be connected, graph-membered sets. Observe that $X \cup Y$ is connected and $X \dot{-} Y$ is connected.

Let X be a tree-like, graph-membered set and Y be a set. Note that $X \cap Y$ is tree-like and $X \setminus Y$ is tree-like.

Let X, Y be tree-like, graph-membered sets. Observe that $X \cup Y$ is tree-like and $X \dot{-} Y$ is tree-like.

Let X be a chordal, graph-membered set and Y be a set. Note that $X \cap Y$ is chordal and $X \setminus Y$ is chordal.

Let X, Y be chordal, graph-membered sets. Observe that $X \cup Y$ is chordal and $X \dot{-} Y$ is chordal.

Let X be an edgeless, graph-membered set and Y be a set. Note that $X \cap Y$ is edgeless and $X \setminus Y$ is edgeless.

Let X, Y be edgeless, graph-membered sets. Observe that $X \cup Y$ is edgeless and $X \dot{-} Y$ is edgeless.

Let X be a loopfull, graph-membered set and Y be a set. Note that $X \cap Y$ is loopfull and $X \setminus Y$ is loopfull.

Let X, Y be loopfull, graph-membered sets. Observe that $X \cup Y$ is loopfull and $X \dot{-} Y$ is loopfull. There exists a graph-membered set which is empty, plain, loopless, non-multi, non-directed-multi, simple, directed-simple, acyclic, connected, tree-like, chordal, edgeless, and loopfull. There exists a graph-membered set which is non empty, tree-like, acyclic, connected, simple, directed-simple, loopless, non-multi, and non-directed-multi.

There exists a graph-membered set which is non empty, edgeless, and chordal and there exists a graph-membered set which is non empty and loopfull and there exists a graph-membered set which is non empty and plain.

Let S be a non empty, plain, graph-membered set. One can verify that every element of S is plain.

Let S be a non empty, loopless, graph-membered set. Let us observe that every element of S is loopless.

Let S be a non empty, non-multi, graph-membered set. Observe that every element of S is non-multi.

Let S be a non empty, non-directed-multi, graph-membered set. Let us note that every element of S is non-directed-multi.

Let S be a non empty, simple, graph-membered set. Note that every element of S is simple.

Let S be a non empty, directed-simple, graph-membered set. Note that every element of S is directed-simple.

Let S be a non empty, acyclic, graph-membered set. Note that every element of S is acyclic.

Let S be a non empty, connected, graph-membered set. One can check that every element of S is connected.

Let S be a non empty, tree-like, graph-membered set. One can verify that every element of S is tree-like.

Let S be a non empty, chordal, graph-membered set. One can verify that every element of S is chordal.

Let S be a non empty, edgeless, graph-membered set. Let us observe that every element of S is edgeless.

Let S be a non empty, loopfull, graph-membered set. Observe that every element of S is loopfull.

Let S be a graph-membered set. The functors: the vertices of S, the edges of S, the source of S, and the target of S yielding sets are defined by conditions

(Def. 14) for every object V, $V \in$ the vertices of S iff there exists a graph G such that $G \in S$ and $V =$ the vertices of G,

(Def. 15) for every object E, $E \in$ the edges of S iff there exists a graph G such that $G \in S$ and $E =$ the edges of G,

(Def. 16) for every object s, $s \in$ the source of S iff there exists a graph G such that $G \in S$ and $s =$ the source of G,

(Def. 17) for every object t, $t \in$ the target of S iff there exists a graph G such that $G \in S$ and $t =$ the target of G,

respectively. Let S be a non empty, graph-membered set. The functors: the vertices of S, the edges of S, the source of S, and the target of S are defined by terms

(Def. 18) the set of all the vertices of G where G is an element of S,

(Def. 19) the set of all the edges of G where G is an element of S,

(Def. 20) the set of all the source of G where G is an element of S,

(Def. 21) the set of all the target of G where G is an element of S,

respectively. One can verify that $\bigcup$(the vertices of S) is non empty.

Let S be a graph-membered set. Note that the source of S is functional and the target of S is functional.

Let S be an empty, graph-membered set. Let us note that the vertices of S is empty and the edges of S is empty and the source of S is empty and the target of S is empty.

Let S be a non empty, graph-membered set. Let us observe that the vertices of S is non empty and the edges of S is non empty and the source of S is non empty and the target of S is non empty.

Let S be a trivial, graph-membered set. Note that the vertices of S is trivial and the edges of S is trivial and the source of S is trivial and the target of S is trivial.

Now we state the propositions:

(3) Let us consider a graph G. Then

  (i) the vertices of $\{G\} = \{$the vertices of $G\}$, and

  (ii) the edges of $\{G\} = \{$the edges of $G\}$, and

  (iii) the source of $\{G\} = \{$the source of $G\}$, and

  (iv) the target of $\{G\} = \{$the target of $G\}$.

(4) Let us consider graphs G, H. Then

  (i) the vertices of $\{G, H\} = \{$the vertices of $G,$ the vertices of $H\}$, and

  (ii) the edges of $\{G, H\} = \{$the edges of $G,$ the edges of $H\}$, and

  (iii) the source of $\{G, H\} = \{$the source of $G,$ the source of $H\}$, and

  (iv) the target of $\{G, H\} = \{$the target of $G,$ the target of $H\}$.

(5) Let us consider a graph-membered set S. Then

  (i) $\overline{\overline{\alpha}} \subseteq \overline{\overline{S}}$, and

  (ii) $\overline{\overline{\beta}} \subseteq \overline{\overline{S}}$, and

  (iii) $\overline{\overline{\gamma}} \subseteq \overline{\overline{S}}$, and

  (iv) $\overline{\overline{\delta}} \subseteq \overline{\overline{S}}$,

where α is the vertices of S, β is the edges of S, γ is the source of S, and δ is the target of S.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the vertices of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f_1 being a function such that $\operatorname{dom} f_1 = S$ and for every object x such that $x \in S$ holds $\mathcal{P}[x, f_1(x)]$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the edges of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{Q}[x, y]$. Consider f_2 being a function such that $\operatorname{dom} f_2 = S$ and for every object x such that $x \in S$ holds $\mathcal{Q}[x, f_2(x)]$.

Define $\mathcal{R}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the source of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{R}[x, y]$. Consider f_3 being a function such that $\operatorname{dom} f_3 = S$ and for every object x such that $x \in S$ holds $\mathcal{R}[x, f_3(x)]$. Define $\mathcal{T}[\text{object}, \text{object}] \equiv$ there exists a graph G such that $\$_1 = G$ and $\$_2 =$ the target of G. For every object x such that $x \in S$ there exists an object y such that $\mathcal{T}[x, y]$. Consider f_4 being a function such that $\operatorname{dom} f_4 = S$ and for every object x such that $x \in S$ holds $\mathcal{T}[x, f_4(x)]$. □

Let S be a finite, graph-membered set. Let us observe that the vertices of S is finite and the edges of S is finite and the source of S is finite and the target of S is finite.

Let S be an edgeless, graph-membered set. Note that $\bigcup$(the edges of S) is empty.

Let us consider graph-membered sets S_1, S_2. Now we state the propositions:

(6) (i) the vertices of $S_1 \cup S_2 =$ (the vertices of S_1) $\cup$ (the vertices of S_2), and

  (ii) the edges of $S_1 \cup S_2 =$ (the edges of S_1) $\cup$ (the edges of S_2), and

(iii) the source of $S_1 \cup S_2 =$ (the source of S_1) $\cup$ (the source of S_2), and

(iv) the target of $S_1 \cup S_2 =$ (the target of S_1) $\cup$ (the target of S_2).

(7) (i) the vertices of $S_1 \cap S_2 \subseteq$ (the vertices of S_1) $\cap$ (the vertices of S_2), and

(ii) the edges of $S_1 \cap S_2 \subseteq$ (the edges of S_1) $\cap$ (the edges of S_2), and

(iii) the source of $S_1 \cap S_2 \subseteq$ (the source of S_1) $\cap$ (the source of S_2), and

(iv) the target of $S_1 \cap S_2 \subseteq$ (the target of S_1) $\cap$ (the target of S_2).

(8) (i) (the vertices of S_1) $\setminus$ (the vertices of S_2) $\subseteq$ the vertices of $S_1 \setminus S_2$, and

(ii) (the edges of S_1) $\setminus$ (the edges of S_2) $\subseteq$ the edges of $S_1 \setminus S_2$, and

(iii) (the source of S_1) $\setminus$ (the source of S_2) $\subseteq$ the source of $S_1 \setminus S_2$, and

(iv) (the target of S_1) $\setminus$ (the target of S_2) $\subseteq$ the target of $S_1 \setminus S_2$.

(9) (i) (the vertices of S_1)$\dot{-}$(the vertices of S_2) $\subseteq$ the vertices of $S_1 \dot{-} S_2$, and

(ii) (the edges of S_1)$\dot{-}$(the edges of S_2) $\subseteq$ the edges of $S_1 \dot{-} S_2$, and

(iii) (the source of S_1)$\dot{-}$(the source of S_2) $\subseteq$ the source of $S_1 \dot{-} S_2$, and

(iv) (the target of S_1)$\dot{-}$(the target of S_2) $\subseteq$ the target of $S_1 \dot{-} S_2$.

The theorem is a consequence of (8) and (6).

2. Union of Graphs

Let G_1, G_2 be graphs. We say that G_1 tolerates G_2 if and only if

(Def. 22) the source of G_1 tolerates the source of G_2 and the target of G_1 tolerates the target of G_2.

Let us observe that the predicate is reflexive and symmetric.

Let us consider graphs G_1, G_2. Now we state the propositions:

(10) If the edges of G_1 misses the edges of G_2, then G_1 tolerates G_2.

(11) Suppose the source of $G_1 \subseteq$ the source of G_2 and the target of $G_1 \subseteq$ the target of G_2. Then G_1 tolerates G_2.

(12) Let us consider a graph G_1, and subgraphs G_2, G_3 of G_1. Then G_2 tolerates G_3.

(13) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then G_1 tolerates G_2. The theorem is a consequence of (12).

Let us consider graphs G_1, G_2. Now we state the propositions:

(14) If $G_1 \approx G_2$, then G_1 tolerates G_2. The theorem is a consequence of (13).

(15) G_1 tolerates G_2 if and only if for every objects e, v_1, w_1, v_2, w_2 such that e joins v_1 to w_1 in G_1 and e joins v_2 to w_2 in G_2 holds $v_1 = v_2$ and $w_1 = w_2$.

(16) Let us consider a graph G_1, a subset E of the edges of G_1, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 tolerates G_2 if and only if $E \subseteq G_1$.loops(). The theorem is a consequence of (15).

Let S be a graph-membered set. We say that S is $\cup$-tolerating if and only if

(Def. 23) for every graphs G_1, G_2 such that $G_1, G_2 \in S$ holds G_1 tolerates G_2.

Let S be a non empty, graph-membered set. Observe that S is $\cup$-tolerating if and only if the condition (Def. 24) is satisfied.

(Def. 24) for every elements G_1, G_2 of S, G_1 tolerates G_2.

One can verify that every graph-membered set which is empty is also $\cup$-tolerating.

Let G be a graph. Observe that $\{G\}$ is $\cup$-tolerating and there exists a graph-membered set which is non empty and $\cup$-tolerating.

A graph union set is a non empty, $\cup$-tolerating, graph-membered set. Now we state the proposition:

(17) Let us consider graphs G_1, G_2. Then G_1 tolerates G_2 if and only if $\{G_1, G_2\}$ is $\cup$-tolerating.

Let S_1 be a $\cup$-tolerating, graph-membered set and S_2 be a set. Let us note that $S_1 \cap S_2$ is $\cup$-tolerating and $S_1 \setminus S_2$ is $\cup$-tolerating.

Now we state the proposition:

(18) Let us consider graph-membered sets S_1, S_2. Suppose $S_1 \cup S_2$ is $\cup$-tolerating. Then

(i) S_1 is $\cup$-tolerating, and

(ii) S_2 is $\cup$-tolerating.

Let S be a $\cup$-tolerating, graph-membered set. Let us note that the source of S is compatible and the target of S is compatible and $\bigcup$(the source of S) is function-like and relation-like and $\bigcup$(the target of S) is function-like and relation-like and $\bigcup$(the source of S) is ($\bigcup$(the edges of S))-defined and ($\bigcup$(the vertices of S))-valued and $\bigcup$(the target of S) is ($\bigcup$(the edges of S))-defined and ($\bigcup$(the vertices of S))-valued and $\bigcup$(the source of S) is total and $\bigcup$(the target of S) is total.

Let S be a graph union set.

A graph union of S is a graph defined by

(Def. 25) the vertices of *it* $= \bigcup$(the vertices of S) and the edges of *it* $= \bigcup$(the edges of S) and the source of *it* $= \bigcup$(the source of S) and the target of *it* $= \bigcup$(the target of S).

Now we state the propositions:

(19) Let us consider a graph union set S, and a graph union G of S. Then every element of S is a subgraph of G.

(20) Let us consider a graph union set S, a graph union G of S, and a graph G'. Then G' is a graph union of S if and only if $G \approx G'$.

Let S be a graph union set. One can check that there exists a graph union of S which is plain and there exists a graph union set which is loopless and there exists a graph union set which is edgeless and there exists a graph union set which is loopfull.

Let S be a loopless graph union set. Note that every graph union of S is loopless.

Let S be an edgeless graph union set. Observe that every graph union of S is edgeless.

Let S be a loopfull graph union set. One can check that every graph union of S is loopfull.

Now we state the proposition:

(21) Let us consider graphs G, H. Then G is a graph union of $\{H\}$ if and only if $G \approx H$. The theorem is a consequence of (3).

Let G_1, G_2 be graphs.

A graph union of G_1 and G_2 is a supergraph of G_1 defined by

(Def. 26) (i) there exists a graph union set S such that $S = \{G_1, G_2\}$ and *it* is a graph union of S, **if** G_1 tolerates G_2,

(ii) *it* $\approx G_1$, **otherwise**.

Now we state the proposition:

(22) Let us consider graphs G_1, G_2, G. Suppose G_1 tolerates G_2. Then G is a graph union of G_1 and G_2 if and only if the vertices of $G =$ (the vertices of G_1) $\cup$ (the vertices of G_2) and the edges of $G =$ (the edges of G_1) $\cup$ (the edges of G_2) and the source of $G =$ (the source of G_1)$+\cdot$(the source of G_2) and the target of $G =$ (the target of G_1)$+\cdot$(the target of G_2). The theorem is a consequence of (4) and (17).

Let us consider graphs G_1, G_2 and a graph union G of G_1 and G_2. Now we state the propositions:

(23) If G_1 tolerates G_2, then G is a supergraph of G_2. The theorem is a consequence of (19).

(24) If G_1 tolerates G_2, then G is a graph union of G_2 and G_1. The theorem is a consequence of (23).

(25) Let us consider graphs G_1, G_2, G', and a graph union G of G_1 and G_2. Then G' is a graph union of G_1 and G_2 if and only if $G \approx G'$. The theorem

is a consequence of (20).

Let G_1, G_2 be graphs. One can verify that there exists a graph union of G_1 and G_2 which is plain.

Now we state the proposition:

(26) Let us consider graphs G, G_1, and a subgraph G_2 of G_1. Then G is a graph union of G_1 and G_2 if and only if $G \approx G_1$. The theorem is a consequence of (13) and (22).

Let G_1, G_2 be loopless graphs. Observe that every graph union of G_1 and G_2 is loopless.

Let G_1, G_2 be edgeless graphs. Let us note that every graph union of G_1 and G_2 is edgeless.

Let G_1, G_2 be loopfull graphs. Note that every graph union of G_1 and G_2 is loopfull.

Now we state the proposition:

(27) Let us consider a graph G_1, a directed graph complement G_2 of G_1 with loops, a graph union G of G_1 and G_2, and vertices v, w of G. Then there exists an object e such that e joins v to w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a directed graph complement of G_1 with loops. Let us observe that every graph union of G_1 and G_2 is loopfull and complete.

Now we state the proposition:

(28) Let us consider a graph G_1, an undirected graph complement G_2 of G_1 with loops, a graph union G of G_1 and G_2, and vertices v, w of G. Then there exists an object e such that e joins v and w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be an undirected graph complement of G_1 with loops. Let us note that every graph union of G_1 and G_2 is loopfull and complete.

Now we state the proposition:

(29) Let us consider a graph G_1, a directed graph complement G_2 of G_1, a graph union G of G_1 and G_2, and vertices v, w of G. If $v \neq w$, then there exists an object e such that e joins v to w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a directed graph complement of G_1. One can check that every graph union of G_1 and G_2 is complete.

Now we state the proposition:

(30) Let us consider a graph G_1, a graph complement G_2 of G_1, a graph union G of G_1 and G_2, and vertices v, w of G. If $v \neq w$, then there exists an object e such that e joins v and w in G. The theorem is a consequence of (10), (22), and (23).

Let G_1 be a graph and G_2 be a graph complement of G_1. Let us note that every graph union of G_1 and G_2 is complete.

Let G_1 be a non-directed-multi graph and G_2 be a directed graph complement of G_1 with loops. One can verify that every graph union of G_1 and G_2 is non-directed-multi.

Let G_1 be a non-multi graph and G_2 be an undirected graph complement of G_1 with loops. Note that every graph union of G_1 and G_2 is non-multi.

Let G_1 be a non-directed-multi graph and G_2 be a directed graph complement of G_1. Observe that every graph union of G_1 and G_2 is non-directed-multi.

Let G_1 be a non-multi graph and G_2 be a graph complement of G_1. One can verify that every graph union of G_1 and G_2 is non-multi.

3. Intersection of Graphs

Let S be a graph-membered set. We say that S is $\cap$-tolerating if and only if

(Def. 27) $\bigcap(\text{the vertices of } S) \neq \emptyset$ and for every graphs G_1, G_2 such that G_1, $G_2 \in S$ holds G_1 tolerates G_2.

Let S be a non empty, graph-membered set. One can verify that S is $\cap$-tolerating if and only if the condition (Def. 28) is satisfied.

(Def. 28) $\bigcap(\text{the vertices of } S) \neq \emptyset$ and for every elements G_1, G_2 of S, G_1 tolerates G_2.

Now we state the proposition:

(31) Let us consider a graph-membered set S. Then S is $\cap$-tolerating if and only if S is $\cup$-tolerating and $\bigcap(\text{the vertices of } S) \neq \emptyset$.

Let G be a graph. Observe that $\{G\}$ is $\cap$-tolerating and every graph-membered set which is $\cap$-tolerating is also $\cup$-tolerating and non empty and there exists a graph-membered set which is $\cap$-tolerating.

A graph meet set is a $\cap$-tolerating, graph-membered set. Let S be a graph meet set. Note that $\bigcap(\text{the vertices of } S)$ is non empty.

Now we state the propositions:

(32) Let us consider graphs G_1, G_2. Then G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2 if and only if $\{G_1, G_2\}$ is $\cap$-tolerating. The theorem is a consequence of (4) and (17).

(33) Let us consider non empty, graph-membered sets S_1, S_2. Suppose $S_1 \cup S_2$ is $\cap$-tolerating. Then

(i) S_1 is $\cap$-tolerating, and

(ii) S_2 is $\cap$-tolerating.

The theorem is a consequence of (6) and (18).

Let S be a graph meet set. One can verify that $\bigcap$(the source of S) is function-like and relation-like and $\bigcap$(the target of S) is function-like and relation-like and $\bigcap$(the source of S) is ($\bigcap$(the edges of S))-defined and ($\bigcap$(the vertices of S))-valued and $\bigcap$(the target of S) is ($\bigcap$(the edges of S))-defined and ($\bigcap$(the vertices of S))-valued and $\bigcap$(the source of S) is total and $\bigcap$(the target of S) is total.

A graph meet of S is a graph defined by

(Def. 29) the vertices of *it* $= \bigcap$(the vertices of S) and the edges of *it* $= \bigcap$(the edges of S) and the source of *it* $= \bigcap$(the source of S) and the target of *it* $= \bigcap$(the target of S).

Now we state the propositions:

(34) Let us consider a graph meet set S, and a graph meet G of S. Then every element of S is a supergraph of G.

(35) Let us consider a graph meet set S, a graph meet G of S, and a graph G'. Then G' is a graph meet of S if and only if $G \approx G'$.

Let S be a graph meet set. Let us observe that there exists a graph meet of S which is plain.

Now we state the proposition:

(36) Let us consider graphs G, H. Then G is a graph meet of $\{H\}$ if and only if $G \approx H$. The theorem is a consequence of (3).

Let G_1, G_2 be graphs.

A graph meet of G_1 and G_2 is a subgraph of G_1 defined by

(Def. 30) (i) there exists a graph meet set S such that $S = \{G_1, G_2\}$ and *it* is a graph meet of S, **if** G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2,

(ii) *it* $\approx G_1$, **otherwise**.

Now we state the proposition:

(37) Let us consider graphs G_1, G_2, G. Suppose G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2. Then G is a graph meet of G_1 and G_2 if and only if the vertices of $G =$ (the vertices of G_1) $\cap$ (the vertices of G_2) and the edges of $G =$ (the edges of G_1) $\cap$ (the edges of G_2) and the source of $G =$ (the source of G_1) $\cap$ (the source of G_2) and the target of $G =$ (the target of G_1) $\cap$ (the target of G_2). The theorem is a consequence of (4) and (32).

Let us consider graphs G_1, G_2 and a graph meet G of G_1 and G_2. Now we state the propositions:

(38) If G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2, then G is a subgraph of G_2. The theorem is a consequence of (34).

(39) If G_1 tolerates G_2 and the vertices of G_1 meets the vertices of G_2, then G is a graph meet of G_2 and G_1. The theorem is a consequence of (38).

(40) Let us consider graphs G_1, G_2, G', and a graph meet G of G_1 and G_2. Then G' is a graph meet of G_1 and G_2 if and only if $G \approx G'$. The theorem is a consequence of (35).

Let G_1, G_2 be graphs. One can check that there exists a graph meet of G_1 and G_2 which is plain.

Now we state the propositions:

(41) Let us consider graphs G, G_1, and a subgraph G_2 of G_1. Then G is a graph meet of G_1 and G_2 if and only if $G \approx G_2$. The theorem is a consequence of (13) and (37).

(42) Let us consider graphs G_1, G_2, and a graph meet G of G_1 and G_2. Suppose the vertices of G_1 meets the vertices of G_2 and the edges of G_1 misses the edges of G_2. Then G is edgeless. The theorem is a consequence of (10) and (37).

Let G_1 be a graph and G_2 be a directed graph complement of G_1 with loops. Let us observe that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be an undirected graph complement of G_1 with loops. One can check that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be a directed graph complement of G_1. Let us note that every graph meet of G_1 and G_2 is edgeless.

Let G_2 be a graph complement of G_1. Let us observe that every graph meet of G_1 and G_2 is edgeless.

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