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Refined Finiteness and Degree Properties in Graphs

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Summary. In this article the finiteness of graphs is refined and the minimal and maximal degree of graphs are formalized in the Mizar system [3], based on the formalization of graphs in [4].

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0. Introduction

The first section introduces the attributes `vertex-finite` and `edge-finite`, which are a refinement of [4]'s `finite`. A notable result is the upper bound of the size of certain graphs in terms of their order, e.g. that a simple finite graph with order n and size m satisfies $m \leqslant \binom{n}{2}$.

Parametrized attributes for the order and size of a graph are introduced in the following section. The main purpose of this additional notation (e.g. `G is n-vertex` instead of `G.order() = n`) is to be used in clusterings and reservations in the future for easy access, e.g. `reserve K2 for simple complete 2-vertex _Graph`.

The third section formalizes locally finite graphs, which are well known (cf. [2], [5], [1]).

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The minimal and maximal degree of a graph are usually defined, together with the degree of a vertex, right at the beginning of general graph theory textbooks, often followed by the Handshaking lemma (cf. [1], [2], [7], [6]). While the Handshaking lemma is still not proven in this article, the last section introduces the minimal and supremal degree of a graph, the latter being called the maximal degree if a vertex attaining the supremal degree exists. This doesn't always have to be the case, of course: Take for example the sum of all complete graphs $\sum_{n=1}^{\infty} K_n$. Therefore the property of a graph having a maximal degree is formalized, too. All formalizations are done as well for in/out degrees and the relationship between them and the undirected degrees is taken into account.

1. Upper Size of Graphs without Parallel Edges

Let us consider a non-directed-multi graph G. Now we state the propositions:

(1) There exists a one-to-one function f such that

 (i) $\operatorname{dom} f =$ the edges of G, and

 (ii) $\operatorname{rng} f \subseteq$ (the vertices of G) $\times$ (the vertices of G), and

 (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \langle\!\langle$(the source of $G)(e)$, (the target of $G)(e)\rangle\!\rangle$.

(2) G.size() $\subseteq G$.order() $\cdot$ G.order(). The theorem is a consequence of (1).

(3) Let us consider a directed-simple graph G. Then there exists a one-to-one function f such that

 (i) $\operatorname{dom} f =$ the edges of G, and

 (ii) $\operatorname{rng} f \subseteq$ ((the vertices of G) $\times$ (the vertices of G)) $\setminus$ (id_{α}), and

 (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \langle\!\langle$(the source of $G)(e)$, (the target of $G)(e)\rangle\!\rangle$,

 where α is the vertices of G. The theorem is a consequence of (1).

(4) Let us consider a non-multi graph G. Then there exists a one-to-one function f such that

 (i) $\operatorname{dom} f =$ the edges of G, and

 (ii) $\operatorname{rng} f \subseteq$ 2Set(the vertices of G) $\cup\, S_{\alpha}$, and

 (iii) for every object e such that $e \in \operatorname{dom} f$ holds $f(e) = \{$(the source of $G)(e)$, (the target of $G)(e)\}$,

 where α is the vertices of G.

(5) Let us consider a simple graph G. Then there exists a one-to-one function f such that

(i) dom f = the edges of G, and

(ii) rng $f \subseteq$ 2Set(the vertices of G), and

(iii) for every object e such that $e \in$ dom f holds $f(e) = \{$(the source of G)(e), (the target of G)$(e)\}$.

Proof: Consider f being a one-to-one function such that dom f = the edges of G and rng $f \subseteq$ 2Set(the vertices of G) $\cup S_\alpha$, where α is the vertices of G and for every object e such that $e \in$ dom f holds $f(e) = \{$(the source of G)(e), (the target of G)$(e)\}$. rng $f \cap S_\alpha = \emptyset$, where α is the vertices of G. $\square$

2. Vertex- and Edge-finite Graphs

Let G be a graph. We say that G is vertex-finite if and only if

(Def. 1) the vertices of G is finite.

We say that G is edge-finite if and only if

(Def. 2) the edges of G is finite.

Let us consider a graph G. Now we state the propositions:

(6) G is vertex-finite if and only if G.order() is finite.

(7) G is edge-finite if and only if G.size() is finite.

(8) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) if G_1 is vertex-finite, then G_2 is vertex-finite, and

(ii) if G_1 is edge-finite, then G_2 is edge-finite.

Let V be a non empty, finite set, E be a set, and S, T be functions from E into V. Observe that createGraph(V, E, S, T) is vertex-finite.

Let V be an infinite set. Let us observe that createGraph(V, E, S, T) is non vertex-finite.

Let V be a non empty set and E be a finite set. Let us observe that createGraph(V, E, S, T) is edge-finite.

Let E be an infinite set. One can verify that createGraph(V, E, S, T) is non edge-finite and every graph which is finite is also vertex-finite and edge-finite and every graph which is vertex-finite and edge-finite is also finite and every graph which is edgeless is also edge-finite and every graph which is trivial is also vertex-finite and every graph which is vertex-finite and non-directed-multi is also edge-finite and every graph which is non vertex-finite and loopfull is also non edge-finite and there exists a graph which is vertex-finite, edge-finite, and simple and there exists a graph which is vertex-finite and non edge-finite and there exists a graph which is non vertex-finite and edge-finite and there exists a graph which is non vertex-finite and non edge-finite.

Let G be a vertex-finite graph. Let us observe that G.order() is non zero and natural.

Let us observe that the functor G.order() yields a non zero natural number. Let G be an edge-finite graph. Let us note that G.size() is natural.

Now we state the propositions:

(9) Let us consider a vertex-finite, non-directed-multi graph G. Then $G.\mathrm{size}() \leqslant (G.\mathrm{order}())^{\mathbf{2}}$. The theorem is a consequence of (2).

(10) Let us consider a vertex-finite, directed-simple graph G. Then $G.\mathrm{size}() \leqslant (G.\mathrm{order}())^{\mathbf{2}} - G.\mathrm{order}()$. The theorem is a consequence of (3).

(11) Let us consider a vertex-finite, non-multi graph G. Then $G.\mathrm{size}() \leqslant \frac{(G.\mathrm{order}())^{\mathbf{2}}+G.\mathrm{order}()}{2}$. The theorem is a consequence of (4).

(12) Let us consider a vertex-finite, simple graph G. Then $G.\mathrm{size}() \leqslant \frac{(G.\mathrm{order}())^{\mathbf{2}}-G.\mathrm{order}()}{2}$. The theorem is a consequence of (5).

Let G be a vertex-finite graph. One can verify that the vertices of G is finite and every subgraph of G is vertex-finite and every directed graph complement of G with loops is vertex-finite and edge-finite and every undirected graph complement of G with loops is vertex-finite and edge-finite and every directed graph complement of G is vertex-finite and edge-finite and every graph complement of G is vertex-finite and edge-finite.

Let V be a finite set. One can check that every supergraph of G extended by the vertices from V is vertex-finite.

Let v be an object. One can check that every supergraph of G extended by v is vertex-finite.

Let e, w be objects. Note that every supergraph of G extended by e between vertices v and w is vertex-finite and every supergraph of G extended by v, w and e between them is vertex-finite.

Let E be a set. One can check that every graph given by reversing directions of the edges E of G is vertex-finite.

Let v be an object and V be a set. Note that every supergraph of G extended by vertex v and edges between v and V of G is vertex-finite and every graph by adding a loop to each vertex of G in V is vertex-finite.

Let G be a graph and V be an infinite set. One can verify that every supergraph of G extended by the vertices from V is non vertex-finite.

Let G be a non vertex-finite graph. Observe that the vertices of G is infinite and every supergraph of G is non vertex-finite and every subgraph of G which is spanning is also non vertex-finite and every directed graph complement of G with loops is non vertex-finite and every undirected graph complement of G

with loops is non vertex-finite and every directed graph complement of G is non vertex-finite and every graph complement of G is non vertex-finite.

Let E be a set. Let us note that every subgraph of G induced by V and E is non vertex-finite.

Let V be an infinite subset of the vertices of G. Note that every graph by adding a loop to each vertex of G in V is non edge-finite.

Let G be an edge-finite graph. One can check that the edges of G is finite and every subgraph of G is edge-finite.

Let V be a set. Note that every supergraph of G extended by the vertices from V is edge-finite.

Let E be a set. Note that every graph given by reversing directions of the edges E of G is edge-finite.

Let v be an object. Note that every supergraph of G extended by v is edge-finite.

Let e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is edge-finite and every supergraph of G extended by v, w and e between them is edge-finite.

Let V be a finite set. Note that every supergraph of G extended by vertex v and edges between v and V of G is edge-finite.

Let V be a finite subset of the vertices of G. Observe that every graph by adding a loop to each vertex of G in V is edge-finite.

Let G be a non vertex-finite, edge-finite graph. Let us observe that there exists a vertex of G which is isolated and every directed graph complement of G with loops is non edge-finite and every undirected graph complement of G with loops is non edge-finite and every directed graph complement of G is non edge-finite and every graph complement of G is non edge-finite.

Let G be a non edge-finite graph. One can verify that the edges of G is infinite and every supergraph of G is non edge-finite.

Let V be a set and E be an infinite subset of the edges of G. Let us observe that every subgraph of G induced by V and E is non edge-finite.

Let E be a finite set. One can verify that every subgraph of G with edges E removed is non edge-finite.

Let e be a set. Let us observe that every subgraph of G with edge e removed is non edge-finite.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(13) Suppose F is weak subgraph embedding. Then

  (i) if G_2 is vertex-finite, then G_1 is vertex-finite, and

  (ii) if G_2 is edge-finite, then G_1 is edge-finite.

(14) If F is onto, then if G_1 is vertex-finite, then G_2 is vertex-finite and if G_1 is edge-finite, then G_2 is edge-finite.

(15) If F is isomorphism, then (G_1 is vertex-finite iff G_2 is vertex-finite) and (G_1 is edge-finite iff G_2 is edge-finite).

3. Order and Size of a Graph as Attributes

Let c be a cardinal number and G be a graph. We say that G is c-vertex if and only if

(Def. 3) G.order() $= c$.

We say that G is c-edge if and only if

(Def. 4) G.size() $= c$.

Let us consider a graph G. Now we state the propositions:

(16) G is vertex-finite if and only if there exists a non zero natural number n such that G is n-vertex.

(17) G is edge-finite if and only if there exists a natural number n such that G is n-edge.

Let us consider graphs G_1, G_2 and a cardinal number c. Now we state the propositions:

(18) Suppose the vertices of $G_1 =$ the vertices of G_2. Then if G_1 is c-vertex, then G_2 is c-vertex.

(19) Suppose the edges of $G_1 =$ the edges of G_2. Then if G_1 is c-edge, then G_2 is c-edge.

(20) If $G_1 \approx G_2$, then if G_1 is c-vertex, then G_2 is c-vertex and if G_1 is c-edge, then G_2 is c-edge.

(21) Every graph G is (G.order())-vertex and (G.size())-edge.

Let V be a non empty set, E be a set, and S, T be functions from E into V. Let us observe that createGraph(V, E, S, T) is $\overline{\overline{V}}$-vertex and $\overline{\overline{E}}$-edge.

Let a be a non zero cardinal number and b be a cardinal number. One can verify that there exists a graph which is a-vertex and b-edge.

Let c be a cardinal number. Let us observe that there exists a graph which is trivial and c-edge and every graph is non 0-vertex and every graph which is trivial is also 1-vertex and every graph which is 1-vertex is also trivial.

Let n be a non zero natural number. One can verify that every graph which is n-vertex is also vertex-finite.

Let c be a non zero cardinal number and G be a c-vertex graph. Observe that every subgraph of G which is spanning is also c-vertex and every directed graph complement of G with loops is c-vertex and every undirected graph complement of G with loops is c-vertex and every directed graph complement of G is c-vertex and every graph complement of G is c-vertex.

Let E be a set. One can verify that every graph given by reversing directions of the edges E of G is c-vertex.

Let V be a set. Let us note that every graph by adding a loop to each vertex of G in V is c-vertex.

Let v, e, w be objects. Observe that every supergraph of G extended by e between vertices v and w is c-vertex and every graph which is edgeless is also 0-edge and every graph which is 0-edge is also edgeless.

Let n be a natural number. Note that every graph which is n-edge is also edge-finite.

Let c be a cardinal number, G be a c-edge graph, and E be a set. Note that every graph given by reversing directions of the edges E of G is c-edge.

Let V be a set. Let us observe that every supergraph of G extended by the vertices from V is c-edge.

Now we state the proposition:

(22) Let us consider graphs G_1, G_2, a partial graph mapping F from G_1 to G_2, and a cardinal number c. Suppose F is isomorphism. Then

(i) G_1 is c-vertex iff G_2 is c-vertex, and

(ii) G_1 is c-edge iff G_2 is c-edge.

4. Locally Finite Graphs

Let G be a graph. We say that G is locally-finite if and only if

(Def. 5) for every vertex v of G, v.edgesInOut() is finite.

Now we state the propositions:

(23) Let us consider a graph G. Then G is locally-finite if and only if for every vertex v of G, v.degree() is finite.

(24) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. If G_1 is locally-finite, then G_2 is locally-finite.

Let us consider a graph G. Now we state the propositions:

(25) G is locally-finite if and only if for every vertex v of G, v.edgesIn() is finite and v.edgesOut() is finite.

(26) G is locally-finite if and only if for every vertex v of G, v.inDegree() is finite and v.outDegree() is finite. The theorem is a consequence of (23).

Let us consider a non empty set V, a set E, and functions S, T from E into V. Now we state the propositions:

(27) Suppose for every element v of V, $S^{-1}(\{v\})$ is finite and $T^{-1}(\{v\})$ is finite. Then createGraph(V, E, S, T) is locally-finite. The theorem is a consequence of (25).

(28) Suppose there exists an element v of V such that $S^{-1}(\{v\})$ is infinite or $T^{-1}(\{v\})$ is infinite. Then createGraph(V, E, S, T) is not locally-finite. The theorem is a consequence of (25).

Let G be a non vertex-finite graph and V be an infinite subset of the vertices of G. One can verify that every supergraph of G extended by vertex the vertices of G and edges between the vertices of G and V of G is non locally-finite and every graph which is edge-finite is also locally-finite and there exists a graph which is locally-finite and there exists a graph which is non locally-finite.

Let G be a locally-finite graph. Note that every subgraph of G is locally-finite.

Let X be a finite set. One can check that G.edgesInto(X) is finite and G.edgesOutOf(X) is finite and G.edgesInOut(X) is finite and G.edgesBetween(X) is finite.

Let Y be a finite set. Note that G.edgesBetween(X, Y) is finite and G.edgesDBetween(X, Y) is finite.

Let v be a vertex of G. One can verify that v.edgesIn() is finite and v.edgesOut() is finite and v.edgesInOut() is finite and v.inDegree() is finite and v.outDegree() is finite and v.degree() is finite.

The functors: v.inDegree(), v.outDegree(), and v.degree() yield natural numbers. Let V be a set. Let us observe that every supergraph of G extended by the vertices from V is locally-finite and every graph by adding a loop to each vertex of G in V is locally-finite.

Let E be a set. Let us observe that every graph given by reversing directions of the edges E of G is locally-finite.

Let v, e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is locally-finite and every supergraph of G extended by v, w and e between them is locally-finite.

Now we state the proposition:

(29) Let us consider a graph G_2, an object v, a subset V of the vertices of G_2, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $v \notin$ the vertices of G_2. Then G_2 is locally-finite and V is finite if and only if G_1 is locally-finite. The theorem is a consequence of (23).

Let G be a locally-finite graph, v be an object, and V be a finite set. Let us note that every supergraph of G extended by vertex v and edges between v and V of G is locally-finite.

Let G be a non locally-finite graph. Let us observe that every supergraph of G is non locally-finite.

Let E be a finite set. Let us note that every subgraph of G with edges E removed is non locally-finite.

Let e be a set. Let us observe that every subgraph of G with edge e removed is non locally-finite.

Now we state the propositions:

(30) Let us consider a non locally-finite graph G_1, a finite subset V of the vertices of G_1, and a subgraph G_2 of G_1 with vertices V removed. Suppose for every vertex v of G_1 such that $v \in V$ holds v.edgesInOut() is finite. Then G_2 is not locally-finite. The theorem is a consequence of (24).

(31) Let us consider a non locally-finite graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. If v.edgesInOut() is finite, then G_2 is not locally-finite. The theorem is a consequence of (30).

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(32) If F is weak subgraph embedding and G_2 is locally-finite, then G_1 is locally-finite. The theorem is a consequence of (23).

(33) If F is onto and semi-directed-continuous and G_1 is locally-finite, then G_2 is locally-finite. The theorem is a consequence of (23).

(34) If F is isomorphism, then G_1 is locally-finite iff G_2 is locally-finite. The theorem is a consequence of (23) and (32).

5. Degree Properties in Graphs

Let G be a graph. The functors: $\bar{\Delta}(G)$, $\bar{\Delta}^-(G)$, $\bar{\Delta}^+(G)$, $\delta(G)$, $\delta^-(G)$, and $\delta^+(G)$ yielding cardinal numbers are defined by terms

(Def. 6) $\bigcup$ the set of all v.degree() where v is a vertex of G.

(Def. 7) $\bigcup$ the set of all v.inDegree() where v is a vertex of G,

(Def. 8) $\bigcup$ the set of all v.outDegree() where v is a vertex of G,

(Def. 9) $\bigcap$ the set of all v.degree() where v is a vertex of G,

(Def. 10) $\bigcap$ the set of all v.inDegree() where v is a vertex of G,

(Def. 11) $\bigcap$ the set of all v.outDegree() where v is a vertex of G,

respectively. Now we state the proposition:

(35) Let us consider a graph G, and a vertex v of G. Then

(i) $\delta(G) \subseteq v$.degree() $\subseteq \bar{\Delta}(G)$, and

(ii) $\delta^-(G) \subseteq v$.inDegree() $\subseteq \bar{\Delta}^-(G)$, and

(iii) $\delta^+(G) \subseteq v.\text{outDegree}() \subseteq \bar{\Delta}^+(G)$.

Let us consider a graph G and a cardinal number c. Now we state the propositions:

(36) $\delta(G) = c$ if and only if there exists a vertex v of G such that $v.\text{degree}() = c$ and for every vertex w of G, $v.\text{degree}() \subseteq w.\text{degree}()$.

(37) $\delta^-(G) = c$ if and only if there exists a vertex v of G such that $v.\text{inDegree}() = c$ and for every vertex w of G, $v.\text{inDegree}() \subseteq w.\text{inDegree}()$.

(38) $\delta^+(G) = c$ if and only if there exists a vertex v of G such that $v.\text{outDegree}() = c$ and for every vertex w of G, $v.\text{outDegree}() \subseteq w.\text{outDegree}()$.

Let us consider a graph G. Now we state the propositions:

(39) $\bar{\Delta}^-(G) \subseteq \bar{\Delta}(G)$.

(40) $\bar{\Delta}^+(G) \subseteq \bar{\Delta}(G)$.

(41) $\delta^-(G) \subseteq \delta(G)$. The theorem is a consequence of (37) and (36).

(42) $\delta^+(G) \subseteq \delta(G)$. The theorem is a consequence of (38) and (36).

(43) $\delta(G) \subseteq \bar{\Delta}(G)$.

(44) $\delta^-(G) \subseteq \bar{\Delta}^-(G)$.

(45) $\delta^+(G) \subseteq \bar{\Delta}^+(G)$.

(46) If there exists a vertex v of G such that v is isolated, then $\delta(G) = 0$ and $\delta^-(G) = 0$ and $\delta^+(G) = 0$. The theorem is a consequence of (36), (37), and (38).

(47) If $\delta(G) = 0$, then there exists a vertex v of G such that v is isolated. The theorem is a consequence of (36).

Let us consider a graph G and a cardinal number c. Now we state the propositions:

(48) If there exists a vertex v of G such that $v.\text{degree}() = c$ and for every vertex w of G, $w.\text{degree}() \subseteq v.\text{degree}()$, then $\bar{\Delta}(G) = c$.

(49) If there exists a vertex v of G such that $v.\text{inDegree}() = c$ and for every vertex w of G, $w.\text{inDegree}() \subseteq v.\text{inDegree}()$, then $\bar{\Delta}^-(G) = c$.

(50) If there exists a vertex v of G such that $v.\text{outDegree}() = c$ and for every vertex w of G, $w.\text{outDegree}() \subseteq v.\text{outDegree}()$, then $\bar{\Delta}^+(G) = c$.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(51) If F is weak subgraph embedding, then $\bar{\Delta}(G_1) \subseteq \bar{\Delta}(G_2)$.

(52) If F is weak subgraph embedding and $\text{rng}\, F_{\mathbb{V}} =$ the vertices of G_2, then $\delta(G_1) \subseteq \delta(G_2)$. The theorem is a consequence of (36).

(53) If F is onto and semi-directed-continuous, then $\bar{\Delta}(G_2) \subseteq \bar{\Delta}(G_1)$.

(54) Suppose F is onto and semi-directed-continuous and $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1. Then $\delta(G_2) \subseteq \delta(G_1)$. The theorem is a consequence of (36).

(55) If F is isomorphism, then $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$ and $\delta(G_1) = \delta(G_2)$. The theorem is a consequence of (51) and (52).

(56) If F is directed and weak subgraph embedding, then $\bar{\Delta}^-(G_1) \subseteq \bar{\Delta}^-(G_2)$ and $\bar{\Delta}^+(G_1) \subseteq \bar{\Delta}^+(G_2)$.

(57) Suppose F is directed and weak subgraph embedding and $\mathrm{rng}\, F_{\mathbb{V}} =$ the vertices of G_2. Then

(i) $\delta^-(G_1) \subseteq \delta^-(G_2)$, and

(ii) $\delta^+(G_1) \subseteq \delta^+(G_2)$.

The theorem is a consequence of (37) and (38).

(58) If F is onto and semi-directed-continuous, then $\bar{\Delta}^-(G_2) \subseteq \bar{\Delta}^-(G_1)$ and $\bar{\Delta}^+(G_2) \subseteq \bar{\Delta}^+(G_1)$.

(59) Suppose F is onto and semi-directed-continuous and $\mathrm{dom}(F_{\mathbb{V}}) =$ the vertices of G_1. Then

(i) $\delta^-(G_2) \subseteq \delta^-(G_1)$, and

(ii) $\delta^+(G_2) \subseteq \delta^+(G_1)$.

The theorem is a consequence of (37) and (38).

(60) Suppose F is directed-isomorphism. Then

(i) $\bar{\Delta}^-(G_1) = \bar{\Delta}^-(G_2)$, and

(ii) $\bar{\Delta}^+(G_1) = \bar{\Delta}^+(G_2)$, and

(iii) $\delta^-(G_1) = \delta^-(G_2)$, and

(iv) $\delta^+(G_1) = \delta^+(G_2)$.

The theorem is a consequence of (56), (57), (58), and (59).

(61) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then

(i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\delta(G_1) = \delta(G_2)$.

(62) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\delta(G_1) = \delta(G_2)$, and

(iii) $\bar{\Delta}^-(G_1) = \bar{\Delta}^-(G_2)$, and

(iv) $\delta^-(G_1) = \delta^-(G_2)$, and

(v) $\bar{\Delta}^{+}(G_1) = \bar{\Delta}^{+}(G_2)$, and

(vi) $\delta^{+}(G_1) = \delta^{+}(G_2)$.

(63) Let us consider a graph G_1, and a subgraph G_2 of G_1. Then

(i) $\bar{\Delta}(G_2) \subseteq \bar{\Delta}(G_1)$, and

(ii) $\bar{\Delta}^{-}(G_2) \subseteq \bar{\Delta}^{-}(G_1)$, and

(iii) $\bar{\Delta}^{+}(G_2) \subseteq \bar{\Delta}^{+}(G_1)$.

The theorem is a consequence of (51) and (56).

(64) Let us consider a graph G_1, and a spanning subgraph G_2 of G_1. Then

(i) $\delta(G_2) \subseteq \delta(G_1)$, and

(ii) $\delta^{-}(G_2) \subseteq \delta^{-}(G_1)$, and

(iii) $\delta^{+}(G_2) \subseteq \delta^{+}(G_1)$.

The theorem is a consequence of (52) and (57).

Let us consider a graph G_2, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Now we state the propositions:

(65) (i) $\bar{\Delta}(G_1) = \bar{\Delta}(G_2)$, and

(ii) $\bar{\Delta}^{-}(G_1) = \bar{\Delta}^{-}(G_2)$, and

(iii) $\bar{\Delta}^{+}(G_1) = \bar{\Delta}^{+}(G_2)$.

The theorem is a consequence of (63).

(66) If $V \setminus$ (the vertices of G_2) $\neq \emptyset$, then $\delta(G_1) = 0$ and $\delta^{-}(G_1) = 0$ and $\delta^{+}(G_1) = 0$. The theorem is a consequence of (46).

Let G be a non edgeless graph. Observe that $\bar{\Delta}(G)$ is non empty and $\bar{\Delta}^{-}(G)$ is non empty and $\bar{\Delta}^{+}(G)$ is non empty.

Let G be a locally-finite graph. One can verify that $\delta(G)$ is natural and $\delta^{-}(G)$ is natural and $\delta^{+}(G)$ is natural.

The functors: $\delta(G)$, $\delta^{-}(G)$, and $\delta^{+}(G)$ yield natural numbers.

Let us consider a locally-finite graph G and a natural number n. Now we state the propositions:

(67) $\delta(G) = n$ if and only if there exists a vertex v of G such that v.degree() $= n$ and for every vertex w of G, v.degree() $\leqslant w$.degree(). The theorem is a consequence of (36).

(68) $\delta^{-}(G) = n$ if and only if there exists a vertex v of G such that v.inDegree() $= n$ and for every vertex w of G, v.inDegree() $\leqslant w$.inDegree(). The theorem is a consequence of (37).

(69) $\delta^{+}(G) = n$ if and only if there exists a vertex v of G such that v.outDegree() $= n$ and for every vertex w of G, v.outDegree() $\leqslant w$.outDegree(). The theorem is a consequence of (38).

Let us consider a graph G_2, vertices v, w of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and w. Now we state the propositions:

(70) If $v \neq w$, then $\delta(G_1) = \delta(G_2)$ or $\delta(G_1) = v.\mathrm{degree}() \cap w.\mathrm{degree}() + 1$. The theorem is a consequence of (36) and (62).

(71) If $v \neq w$, then $\delta^-(G_1) = \delta^-(G_2)$ or $\delta^-(G_1) = w.\mathrm{inDegree}() + 1$. The theorem is a consequence of (37) and (62).

(72) If $v \neq w$, then $\delta^+(G_1) = \delta^+(G_2)$ or $\delta^+(G_1) = v.\mathrm{outDegree}() + 1$. The theorem is a consequence of (38) and (62).

Let us consider a locally-finite graph G_2, vertices v, w of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and w. Now we state the propositions:

(73) If $v \neq w$, then $\delta(G_1) = \delta(G_2)$ or $\delta(G_1) = \min(v.\mathrm{degree}(), w.\mathrm{degree}())+1$. The theorem is a consequence of (70).

(74) If $v \neq w$, then $\delta^-(G_1) = \delta^-(G_2)$ or $\delta^-(G_1) = w.\mathrm{inDegree}() + 1$. The theorem is a consequence of (71).

(75) If $v \neq w$, then $\delta^+(G_1) = \delta^+(G_2)$ or $\delta^+(G_1) = v.\mathrm{outDegree}() + 1$. The theorem is a consequence of (72).

(76) Let us consider a graph G_2, an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and the vertices of G_2. Suppose $v \notin$ the vertices of G_2. Then $\delta(G_1) = (\delta(G_2)+1) \cap G_2.\mathrm{order}()$. The theorem is a consequence of (36).

(77) Let us consider a finite graph G_2, an object v, and a supergraph G_1 of G_2 extended by vertex v and edges between v and the vertices of G_2. Suppose $v \notin$ the vertices of G_2. Then $\delta(G_1) = \min(\delta(G_2)+1, G_2.\mathrm{order}())$. The theorem is a consequence of (76).

(78) Let us consider a graph G_2, a set V, and a graph G_1 by adding a loop to each vertex of G_2 in V. Then $\delta(G_1) \subseteq \delta(G_2) + 2$. The theorem is a consequence of (36) and (62).

Let G be an edge-finite graph. One can check that $\bar{\Delta}(G)$ is natural and $\bar{\Delta}^-(G)$ is natural and $\bar{\Delta}^+(G)$ is natural.

The functors: $\bar{\Delta}(G)$, $\bar{\Delta}^-(G)$, and $\bar{\Delta}^+(G)$ yield natural numbers. Let G be a graph. We say that G is with max degree if and only if

(Def. 12) there exists a vertex v of G such that for every vertex w of G, $w.\mathrm{degree}() \subseteq v.\mathrm{degree}()$.

We say that G is with max indegree if and only if

(Def. 13) there exists a vertex v of G such that for every vertex w of G, $w.\mathrm{inDegree}() \subseteq v.\mathrm{inDegree}()$.

We say that G is with max outdegree if and only if

(Def. 14) there exists a vertex v of G such that for every vertex w of G, w.outDegree() $\subseteq v$.outDegree().

Let us consider a graph G. Now we state the propositions:

(79) If G is with max degree, then there exists a vertex v of G such that

(i) v.degree() $= \bar{\Delta}(G)$, and

(ii) for every vertex w of G, w.degree() $\subseteq v$.degree().

The theorem is a consequence of (35).

(80) Suppose G is with max indegree. Then there exists a vertex v of G such that

(i) v.inDegree() $= \bar{\Delta}^{-}(G)$, and

(ii) for every vertex w of G, w.inDegree() $\subseteq v$.inDegree().

The theorem is a consequence of (35).

(81) Suppose G is with max outdegree. Then there exists a vertex v of G such that

(i) v.outDegree() $= \bar{\Delta}^{+}(G)$, and

(ii) for every vertex w of G, w.outDegree() $\subseteq v$.outDegree().

The theorem is a consequence of (35).

Let G be a graph. We introduce the notation G is without max degree as an antonym for G is with max degree. We introduce the notation G is without max indegree as an antonym for G is with max indegree. We introduce the notation G is without max outdegree as an antonym for G is with max outdegree.

Let us note that every graph which is with max indegree and with max outdegree is also with max degree and every graph which is vertex-finite is also with max degree, with max indegree, and with max outdegree and every graph which is edge-finite is also with max degree, with max indegree, and with max outdegree.

Now we state the proposition:

(82) Every with max degree graph is with max indegree or with max outdegree. The theorem is a consequence of (79), (40), (35), and (39).

Let G be a with max degree graph. We introduce the notation $\Delta(G)$ as a synonym of $\bar{\Delta}(G)$.

Let G be a with max indegree graph. We introduce the notation $\Delta^{-}(G)$ as a synonym of $\bar{\Delta}^{-}(G)$.

Let G be a with max outdegree graph. We introduce the notation $\Delta^{+}(G)$ as a synonym of $\bar{\Delta}^{+}(G)$.

Let G be a locally-finite, with max degree graph. Let us note that $\Delta(G)$ is natural.

Note that the functor $\Delta(G)$ yields a natural number. Let G be a locally-finite, with max indegree graph. Let us note that $\Delta^-(G)$ is natural.

Note that the functor $\Delta^-(G)$ yields a natural number. Let G be a locally-finite, with max outdegree graph. Let us note that $\Delta^+(G)$ is natural.

Note that the functor $\Delta^+(G)$ yields a natural number.

Let us consider graphs G_1, G_2 and a partial graph mapping F from G_1 to G_2. Now we state the propositions:

(83) If F is isomorphism, then G_1 is with max degree iff G_2 is with max degree. The theorem is a consequence of (79) and (55).

(84) Suppose F is directed-isomorphism. Then

(i) G_1 is with max indegree iff G_2 is with max indegree, and

(ii) G_1 is with max outdegree iff G_2 is with max outdegree.

The theorem is a consequence of (80), (60), and (81).

(85) Let us consider graphs G_1, G_2. Suppose $G_1 \approx G_2$. Then

(i) if G_1 is with max degree, then G_2 is with max degree, and

(ii) if G_1 is with max indegree, then G_2 is with max indegree, and

(iii) if G_1 is with max outdegree, then G_2 is with max outdegree.

The theorem is a consequence of (83) and (84).

(86) Let us consider a graph G_1, a set E, and a graph G_2 given by reversing directions of the edges E of G_1. Then G_1 is with max degree if and only if G_2 is with max degree. The theorem is a consequence of (83).

Let G be a with max degree graph and E be a set. Observe that every graph given by reversing directions of the edges E of G is with max degree.

Let V be a set. Let us note that every supergraph of G extended by the vertices from V is with max degree and every graph by adding a loop to each vertex of G in V is with max degree.

Let v, e, w be objects. One can verify that every supergraph of G extended by e between vertices v and w is with max degree and every supergraph of G extended by v, w and e between them is with max degree.

Let v be an object and V be a set. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is with max degree.

Let G be a with max indegree graph. Observe that every graph given by reversing directions of the edges of G is with max outdegree.

Let V be a set. One can verify that every supergraph of G extended by the vertices from V is with max indegree and every graph by adding a loop to each vertex of G in V is with max indegree.

Let v, e, w be objects. Let us note that every supergraph of G extended by e between vertices v and w is with max indegree and every supergraph of G extended by v, w and e between them is with max indegree.

Let v be an object and V be a set. Let us note that every supergraph of G extended by vertex v and edges between v and V of G is with max indegree.

Let G be a with max outdegree graph. One can check that every graph given by reversing directions of the edges of G is with max indegree.

Let V be a set. Let us note that every supergraph of G extended by the vertices from V is with max outdegree and every graph by adding a loop to each vertex of G in V is with max outdegree.

Let v, e, w be objects. One can verify that every supergraph of G extended by e between vertices v and w is with max outdegree and every supergraph of G extended by v, w and e between them is with max outdegree.

Let v be an object and V be a set. One can verify that every supergraph of G extended by vertex v and edges between v and V of G is with max outdegree.

Now we state the propositions:

(87) Let us consider a locally-finite, with max degree graph G, and a natural number n. Then $\Delta(G) = n$ if and only if there exists a vertex v of G such that v.degree() $= n$ and for every vertex w of G, w.degree() $\leqslant v$.degree(). The theorem is a consequence of (79) and (48).

(88) Let us consider a locally-finite, with max indegree graph G, and a natural number n. Then $\Delta^{-}(G) = n$ if and only if there exists a vertex v of G such that v.inDegree() $= n$ and for every vertex w of G, w.inDegree() $\leqslant$ v.inDegree(). The theorem is a consequence of (80) and (49).

(89) Let us consider a locally-finite, with max outdegree graph G, and a natural number n. Then $\Delta^{+}(G) = n$ if and only if there exists a vertex v of G such that v.outDegree() $= n$ and for every vertex w of G, w.outDegree() $\leqslant$ v.outDegree(). The theorem is a consequence of (81) and (50).

(90) Let us consider a cardinal number c, and a trivial, c-edge graph G. Then

(i) $\Delta^{-}(G) = c$, and

(ii) $\delta^{-}(G) = c$, and

(iii) $\Delta^{+}(G) = c$, and

(iv) $\delta^{+}(G) = c$, and

(v) $\Delta(G) = c + c$, and

(vi) $\delta(G) = c + c$.

The theorem is a consequence of (49), (37), (50), (38), (48), and (36).

Let G be a graph and v be a vertex of G. We say that v is with min degree if and only if

(Def. 15) v.degree() $= \delta(G)$.

We say that v is with min indegree if and only if

(Def. 16) v.inDegree() $= \delta^-(G)$.

We say that v is with min outdegree if and only if

(Def. 17) v.outDegree() $= \delta^+(G)$.

We say that v is with max degree if and only if

(Def. 18) v.degree() $= \bar{\Delta}(G)$.

We say that v is with max indegree if and only if

(Def. 19) v.inDegree() $= \bar{\Delta}^-(G)$.

We say that v is with max outdegree if and only if

(Def. 20) v.outDegree() $= \bar{\Delta}^+(G)$.

Let us consider a graph G and vertices v, w of G. Now we state the propositions:

(91) If v is with min degree, then v.degree() $\subseteq w$.degree(). The theorem is a consequence of (36).

(92) If v is with min indegree, then v.inDegree() $\subseteq w$.inDegree(). The theorem is a consequence of (37).

(93) If v is with min outdegree, then v.outDegree() $\subseteq w$.outDegree(). The theorem is a consequence of (38).

(94) If w is with max degree, then v.degree() $\subseteq w$.degree(). The theorem is a consequence of (79).

(95) If w is with max indegree, then v.inDegree() $\subseteq w$.inDegree(). The theorem is a consequence of (80).

(96) If w is with max outdegree, then v.outDegree() $\subseteq w$.outDegree(). The theorem is a consequence of (81).

Let G be a graph. Note that there exists a vertex of G which is with min degree and there exists a vertex of G which is with min indegree and there exists a vertex of G which is with min outdegree and every vertex of G which is with min indegree and with min outdegree is also with min degree and every vertex of G which is with max indegree and with max outdegree is also with max degree and every vertex of G which is isolated is also with min degree, with min indegree, and with min outdegree.

Let us consider a graph G. Now we state the propositions:

(97) G is with max degree if and only if there exists a vertex v of G such that v is with max degree. The theorem is a consequence of (79).

(98) G is with max indegree if and only if there exists a vertex v of G such that v is with max indegree. The theorem is a consequence of (80).

(99) G is with max outdegree if and only if there exists a vertex v of G such that v is with max outdegree. The theorem is a consequence of (81).

Let G be a with max degree graph. Observe that there exists a vertex of G which is with max degree.

Let G be a with max indegree graph. One can check that there exists a vertex of G which is with max indegree.

Let G be a with max outdegree graph. Observe that there exists a vertex of G which is with max outdegree.

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