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Implicit Function Theorem. Part II

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Summary. In this article, we formalize differentiability of implicit function theorem in the Mizar system [3], [1]. In the first half section, properties of Lipschitz continuous linear operators are discussed. Some norm properties of a direct sum decomposition of Lipschitz continuous linear operator are mentioned here.

In the last half section, differentiability of implicit function in implicit function theorem is formalized. The existence and uniqueness of implicit function in [6] is cited. We referred to [10], [11], and [2] in the formalization.

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1. Properties of Lipschitz Continuous Linear Operators

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

(1) Let us consider real normed spaces E, F, a partial function f from E to F, a subset Z of E, and a point z of E. Suppose Z is open and $z \in Z$ and $Z \subseteq \operatorname{dom} f$ and f is differentiable in z. Then

(i) $f{\upharpoonright}Z$ is differentiable in z, and

(ii) $f'(z) = (f{\upharpoonright}Z)'(z)$.

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Proof: Consider N being a neighbourhood of z such that $N \subseteq \operatorname{dom} f$ and there exists a rest R of E, F such that for every point x of E such that $x \in N$ holds $f_{/x} - f_{/z} = (f'(z))(x - z) + R_{/x-z}$. Consider r being a real number such that $r > 0$ and $\operatorname{Ball}(z, r) \subseteq Z$. Reconsider $N_4 = N \cap Z$ as a neighbourhood of z. Consider R being a rest of E, F such that for every point x of E such that $x \in N$ holds $f_{/x} - f_{/z} = (f'(z))(x - z) + R_{/x-z}$. For every point x of E such that $x \in N_4$ holds $(f \restriction Z)_{/x} - (f \restriction Z)_{/z} = (f'(z))(x - z) + R_{/x-z}$. □

(2) Let us consider real normed spaces E, F, G, a partial function f from $E \times F$ to G, a subset Z of $E \times F$, and a point z of $E \times F$. Suppose Z is open and $z \in Z$ and $Z \subseteq \operatorname{dom} f$. Then

(i) if f is partially differentiable in z w.r.t. 1, then $f \restriction Z$ is partially differentiable in z w.r.t. 1 and $\operatorname{partdiff}(f, z)$ w.r.t. $1 =$ $\operatorname{partdiff}(f \restriction Z, z)$ w.r.t. 1, and

(ii) if f is partially differentiable in z w.r.t. 2, then $f \restriction Z$ is partially differentiable in z w.r.t. 2 and $\operatorname{partdiff}(f, z)$ w.r.t. $2 =$ $\operatorname{partdiff}(f \restriction Z, z)$ w.r.t. 2.

Proof: If f is partially differentiable in z w.r.t. 1, then $f \restriction Z$ is partially differentiable in z w.r.t. 1 and $\operatorname{partdiff}(f, z)$ w.r.t. $1 = \operatorname{partdiff}(f \restriction Z, z)$ w.r.t. 1. Set $g = f \cdot (\operatorname{reproj2}(z))$. Consider N being a neighbourhood of $(z)_{\mathbf{2}}$ such that $N \subseteq \operatorname{dom} g$ and there exists a rest R of F, G such that for every point x of F such that $x \in N$ holds $g_{/x} - g_{/(z)_{\mathbf{2}}} = (\operatorname{partdiff}(f, z)$ w.r.t. $2)(x - (z)_{\mathbf{2}}) + R_{/x-(z)_{\mathbf{2}}}$. Consider R being a rest of F, G such that for every point x of F such that $x \in N$ holds $g_{/x} - g_{/(z)_{\mathbf{2}}} = (\operatorname{partdiff}(f, z)$ w.r.t. $2)(x - (z)_{\mathbf{2}}) + R_{/x-(z)_{\mathbf{2}}}$.

Set $h = (f \restriction Z) \cdot (\operatorname{reproj2}(z))$. Consider r_1 being a real number such that $r_1 > 0$ and $\operatorname{Ball}(z, r_1) \subseteq Z$. Consider r_2 being a real number such that $r_2 > 0$ and $\{y$, where y is a point of $F : \|y - (z)_{\mathbf{2}}\| < r_2\} \subseteq N$. Set $r = \min(r_1, r_2)$. Set $M = \operatorname{Ball}((z)_{\mathbf{2}}, r)$. $M \subseteq N$ and for every point x of F such that $x \in M$ holds $(\operatorname{reproj2}(z))(x) \in Z$. $M \subseteq \operatorname{dom} h$. For every point x of F such that $x \in M$ holds $h_{/x} - h_{/(z)_{\mathbf{2}}} = (\operatorname{partdiff}(f, z)$ w.r.t. $2)(x - (z)_{\mathbf{2}}) + R_{/x-(z)_{\mathbf{2}}}$. □

(3) Let us consider real normed spaces X, E, G, F, a bilinear operator B from $E \times F$ into G, a partial function f from X to E, a partial function g from X to F, and a subset S of X. Suppose B is continuous on the carrier of $E \times F$ and $S \subseteq \operatorname{dom} f$ and $S \subseteq \operatorname{dom} g$ and for every point s of X such that $s \in S$ holds f is continuous in s and for every point s of X such that $s \in S$ holds g is continuous in s. Then there exists a partial function H from X to G such that

(i) $\operatorname{dom} H = S$, and

(ii) for every point s of X such that $s \in S$ holds $H(s) = B(f(s), g(s))$, and

(iii) H is continuous on S.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point t of X such that $t = \$_1$ and $\$_2 = B(f(t), g(t))$. For every object x such that $x \in S$ there exists an object y such that $y \in$ the carrier of G and $\mathcal{P}[x, y]$. Consider H being a function from S into G such that for every object z such that $z \in S$ holds $\mathcal{P}[z, H(z)]$. For every point s of X such that $s \in S$ holds $H(s) = B(f(s), g(s))$. For every point x_0 of X and for every real number r such that $x_0 \in S$ and $0 < r$ there exists a real number p_2 such that $0 < p_2$ and for every point x_1 of X such that $x_1 \in S$ and $\|x_1 - x_0\| < p_2$ holds $\|H_{/x_1} - H_{/x_0}\| < r$. □

(4) Let us consider real normed spaces E, F, a partial function g from E to F, and a subset A of E. Suppose g is continuous on A and $\operatorname{dom} g = A$. Then there exists a partial function x_2 from E to $E \times F$ such that

(i) $\operatorname{dom} x_2 = A$, and

(ii) for every point x of E such that $x \in A$ holds $x_2(x) = \langle x, g(x) \rangle$, and

(iii) x_2 is continuous on A.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point t of E such that $t = \$_1$ and $\$_2 = \langle t, g(t) \rangle$. For every object x such that $x \in S$ there exists an object y such that $y \in$ the carrier of $E \times F$ and $\mathcal{P}[x, y]$. Consider H being a function from S into $E \times F$ such that for every object z such that $z \in S$ holds $\mathcal{P}[z, H(z)]$. For every point s of E such that $s \in S$ holds $H(s) = \langle s, g(s) \rangle$. For every point x_0 of E and for every real number r such that $x_0 \in S$ and $0 < r$ there exists a real number p_2 such that $0 < p_2$ and for every point x_1 of E such that $x_1 \in S$ and $\|x_1 - x_0\| < p_2$ holds $\|H_{/x_1} - H_{/x_0}\| < r$. □

(5) Let us consider real normed spaces S, T, V, a point x_0 of V, a partial function f_1 from the carrier of V to the carrier of S, and a partial function f_2 from the carrier of S to the carrier of T. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in f_{1/x_0}. Then $f_2 \cdot f_1$ is continuous in x_0.

PROOF: $\operatorname{rng}(f_{1*} s_1) \subseteq \operatorname{dom} f_2$. □

(6) Let us consider real normed spaces E, F, a point z of $E \times F$, a point x of E, and a point y of F. Suppose $z = \langle x, y \rangle$. Then $\|z\| \leqslant \|x\| + \|y\|$.

(7) Let us consider real normed spaces E, F, G, and a linear operator L from $E \times F$ into G. Then there exists a linear operator L_1 from E into G

and there exists a linear operator L_2 from F into G such that for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x) + L_2(y)$ and for every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$ and for every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$.

PROOF: Define $\mathcal{C}(\text{point of } E) = L_{/\langle \$_1, 0_F\rangle}$. Consider L_1 being a function from the carrier of E into the carrier of G such that for every point x of E, $L_1(x) = \mathcal{C}(x)$. For every elements s, t of E, $L_1(s+t) = L_1(s) + L_1(t)$. For every element s of E and for every real number r, $L_1(r \cdot s) = r \cdot L_1(s)$. Define $\mathcal{D}(\text{point of } F) = L_{/\langle 0_E, \$_1\rangle}$. Consider L_2 being a function from the carrier of F into the carrier of G such that for every point x of F, $L_2(x) = \mathcal{D}(x)$. For every elements s, t of F, $L_2(s+t) = L_2(s) + L_2(t)$. For every element s of F and for every real number r, $L_2(r \cdot s) = r \cdot L_2(s)$. For every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x) + L_2(y)$. □

(8) Let us consider real normed spaces E, F, G, a linear operator L from $E \times F$ into G, a linear operator L_{11} from E into G, a linear operator L_{12} from F into G, a linear operator L_{21} from E into G, and a linear operator L_{22} from F into G. Suppose for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_{11}(x) + L_{12}(y)$ and for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_{21}(x) + L_{22}(y)$. Then

(i) $L_{11} = L_{21}$, and

(ii) $L_{12} = L_{22}$.

(9) Let us consider real normed spaces E, F, G, a linear operator L_1 from E into G, and a linear operator L_2 from F into G. Then there exists a linear operator L from $E \times F$ into G such that

(i) for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x) + L_2(y)$, and

(ii) for every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$, and

(iii) for every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point x of E and there exists a point y of F such that $\$_1 = \langle x, y\rangle$ and $\$_2 = L_1(x) + L_2(y)$. For every element z of $E \times F$, there exists an element y of G such that $\mathcal{P}[z, y]$. Consider L being a function from $E \times F$ into G such that for every element z of $E \times F$, $\mathcal{P}[z, L(z)]$. For every points z, w of $E \times F$, $L(z+w) = L(z) + L(w)$. For every element z of $E \times F$ and for every real number r, $L(r \cdot z) = r \cdot L(z)$. For every point x of E and for every point y

of F, $L(\langle x, y\rangle) = L_1(x)+L_2(y)$. For every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$. For every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$ by [9, (3)]. □

(10) Let us consider real normed spaces E, F, G, and a Lipschitzian linear operator L from $E \times F$ into G. Then there exists a Lipschitzian linear operator L_1 from E into G and there exists a Lipschitzian linear operator L_2 from F into G such that for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x)+L_2(y)$ and for every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$ and for every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$. The theorem is a consequence of (7).

(11) Let us consider real normed spaces E, F, G, a Lipschitzian linear operator L_1 from E into G, and a Lipschitzian linear operator L_2 from F into G. Then there exists a Lipschitzian linear operator L from $E \times F$ into G such that

(i) for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x)+L_2(y)$, and

(ii) for every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$, and

(iii) for every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$.

The theorem is a consequence of (9).

(12) Let us consider real normed spaces E, F, G, and a point L of the real norm space of bounded linear operators from $E \times F$ into G. Then there exists a point L_1 of the real norm space of bounded linear operators from E into G and there exists a point L_2 of the real norm space of bounded linear operators from F into G such that for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x) + L_2(y)$ and for every point x of E, $L_1(x) = L(\langle x, 0_F\rangle)$ and for every point y of F, $L_2(y) = L(\langle 0_E, y\rangle)$ and $\|L\| \leqslant \|L_1\| + \|L_2\|$ and $\|L_1\| \leqslant \|L\|$ and $\|L_2\| \leqslant \|L\|$.

PROOF: Reconsider $L = L_4$ as a Lipschitzian linear operator from $E \times F$ into G. Consider L_1 being a Lipschitzian linear operator from E into G, L_2 being a Lipschitzian linear operator from F into G such that for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_1(x) + L_2(y)$ and for every point x of E, $L_1(x) = L_{/\langle x, 0_F\rangle}$ and for every point y of F, $L_2(y) = L_{/\langle 0_E, y\rangle}$.

Reconsider $L_5 = L_1$ as a point of the real norm space of bounded linear operators from E into G. Reconsider $L_3 = L_2$ as a point of the real norm space of bounded linear operators from F into G. For every point x of E, $L_5(x) = L_4(\langle x, 0_F\rangle)$. For every point y of F, $L_3(y) = L_4(\langle 0_E, y\rangle)$. For every real number t such that $t \in \mathrm{PreNorms}(L)$ holds $t \leqslant \|L_5\|+\|L_3\|$.

For every real number t such that $t \in \mathrm{PreNorms}(L_1)$ holds $t \leqslant \|L_4\|$. For every real number t such that $t \in \mathrm{PreNorms}(L_2)$ holds $t \leqslant \|L_4\|$. □

(13) Let us consider real normed spaces E, F, G, a point L of the real norm space of bounded linear operators from $E \times F$ into G, points L_{11}, L_{12} of the real norm space of bounded linear operators from E into G, and points L_{21}, L_{22} of the real norm space of bounded linear operators from F into G. Suppose for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_{11}(x) + L_{21}(y)$ and for every point x of E and for every point y of F, $L(\langle x, y\rangle) = L_{12}(x) + L_{22}(y)$. Then

- (i) $L_{11} = L_{12}$, and
- (ii) $L_{21} = L_{22}$.

The theorem is a consequence of (8).

2. Differentiability of Implicit Function

Now we state the propositions:

(14) Let us consider real normed spaces E, G, F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, and a point z of $E \times F$. Suppose f is differentiable in z. Then

- (i) f is partially differentiable in z w.r.t. 1, and
- (ii) f is partially differentiable in z w.r.t. 2, and
- (iii) for every point d_7 of E and for every point d_8 of F, $(f'(z))(\langle d_7, d_8\rangle) = (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1)(d_7) + (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 2)(d_8)$.

Proof: Reconsider $y = (\mathrm{IsoCPNrSP}(E, F))(z)$ as a point of $\prod\langle E, F\rangle$. Consider N being a neighbourhood of z such that $N \subseteq \mathrm{dom}\, f$ and there exists a rest R of $E \times F$, G such that for every point w of $E \times F$ such that $w \in N$ holds $f_{/w} - f_{/z} = (f'(z))(w - z) + R_{/w-z}$. Consider R being a rest of $E \times F$, G such that for every point w of $E \times F$ such that $w \in N$ holds $f_{/w} - f_{/z} = (f'(z))(w - z) + R_{/w-z}$. Reconsider $L = f'(z)$ as a Lipschitzian linear operator from $E \times F$ into G. Consider L_1 being a Lipschitzian linear operator from E into G, L_2 being a Lipschitzian linear operator from F into G such that for every point d_7 of E and for every point d_8 of F, $L(\langle d_7, d_8\rangle) = L_1(d_7) + L_2(d_8)$ and for every point d_7 of E, $L_1(d_7) = L_{/\langle d_7, 0_F\rangle}$ and for every point d_8 of F, $L_2(d_8) = L_{/\langle 0_E, d_8\rangle}$.

Reconsider $L_3 = L_1$ as a point of the real norm space of bounded linear operators from E into G. Reconsider $L_4 = L_2$ as a point of the real norm space of bounded linear operators from F into G. Set $g_1 = f \cdot (\mathrm{reproj1}(z))$.

Set $g_2 = f \cdot (\text{reproj2}(z))$. Reconsider $x = (z)_{\mathbf{1}}$ as a point of E. Reconsider $y = (z)_{\mathbf{2}}$ as a point of F. Consider r_0 being a real number such that $0 < r_0$ and $\{y$, where y is a point of $E \times F : \|y - z\| < r_0\} \subseteq N$. Consider r being a real number such that $0 < r < r_0$ and $\text{Ball}(x, r) \times \text{Ball}(y, r) \subseteq \text{Ball}(z, r_0)$. Define $\mathcal{C}(\text{point of } E) = R_{/\langle \$_1, 0_F \rangle}$. Consider R_1 being a function from the carrier of E into the carrier of G such that for every point p of E, $R_1(p) = \mathcal{C}(p)$. Define $\mathcal{D}(\text{point of } F) = R_{/\langle 0_E, \$_1 \rangle}$. Consider R_2 being a function from the carrier of F into the carrier of G such that for every point p of F, $R_2(p) = \mathcal{D}(p)$.

For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of E such that $z \neq 0_E$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{1/z}\| < r$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of F such that $z \neq 0_F$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{2/z}\| < r$. Reconsider $N_1 = \text{Ball}(x, r)$ as a neighbourhood of x. Reconsider $N_2 = \text{Ball}(y, r)$ as a neighbourhood of y. $N_1 \subseteq \text{dom}\, g_1$. $N_2 \subseteq \text{dom}\, g_2$. For every point x_1 of E such that $x_1 \in N_1$ holds $g_{1/x_1} - g_{1/x} = L_3(x_1 - x) + R_{1/x_1 - x}$. For every point y_1 of F such that $y_1 \in N_2$ holds $g_{2/y_1} - g_{2/y} = L_4(y_1 - y) + R_{2/y_1 - y}$. □

(15) Let us consider real normed spaces E, G, F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point a of E, a point b of F, a point c of G, a point z of $E \times F$, real numbers r_1, r_2, a partial function g from E to F, a Lipschitzian linear operator P from E into G, and a Lipschitzian linear operator Q from G into F.

Suppose Z is open and $\text{dom}\, f = Z$ and $z = \langle a, b \rangle$ and $z \in Z$ and $f(a, b) = c$ and f is differentiable in z and $0 < r_1$ and $0 < r_2$ and $\text{dom}\, g = \text{Ball}(a, r_1)$ and $\text{rng}\, g \subseteq \text{Ball}(b, r_2)$ and $g(a) = b$ and g is continuous in a and for every point x of E such that $x \in \text{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and partdiff(f, z) w.r.t. 2 is one-to-one and $Q = (\text{partdiff}(f, z) \text{ w.r.t. } 2)^{-1}$ and $P = \text{partdiff}(f, z)$ w.r.t. 1. Then

(i) g is differentiable in a, and

(ii) $g'(a) = -Q \cdot P$.

PROOF: Reconsider $L = Q \cdot P$ as a point of the real norm space of bounded linear operators from E into F. Consider N_0 being a neighbourhood of z such that $N_0 \subseteq \text{dom}\, f$ and there exists a rest R of $E \times F$, G such that for every point w of $E \times F$ such that $w \in N_0$ holds $f_{/w} - f_{/z} = (f'(z))(w - z) + R_{/w-z}$. Consider R being a rest of $E \times F$, G such that for every point w of $E \times F$ such that $w \in N_0$ holds $f_{/w} - f_{/z} = (f'(z))(w - z) + R_{/w-z}$. Consider r_0 being a real number such that $0 < r_0$ and $\{y$, where y is a point of $E \times F : \|y - z\| < r_0\} \subseteq N_0$. Consider r_3 being a real number such that $0 < r_3 < r_0$ and $\mathrm{Ball}(a, r_3) \times \mathrm{Ball}(b, r_3) \subseteq \mathrm{Ball}(z, r_0)$. Reconsider $r_4 = \min(r_1, r_3)$ as a real number.

Consider r_5 being a real number such that $0 < r_5$ and for every point x_1 of E such that $x_1 \in \mathrm{dom}\, g$ and $\|x_1 - a\| < r_5$ holds $\|g_{/x_1} - g_{/a}\| < r_3$. Reconsider $r_6 = \min(r_4, r_5)$ as a real number. Reconsider $N = \mathrm{Ball}(a, r_6)$ as a neighbourhood of a. Define $\mathcal{C}(\text{point of } E) = -Q(R_{/\langle \$_1, g_{/a+\$_1} - g_{/a} \rangle})$. Consider R_1 being a function from the carrier of E into the carrier of F such that for every point p of E, $R_1(p) = \mathcal{C}(p)$. For every point x of E such that $x \in N$ holds $g_{/x} - g_{/a} = (-L)(x - a) + R_{1/x-a}$. Define $\mathcal{D}[\text{point of } E, \text{object}] \equiv \$_2 = \langle \$_1, g_{/a+\$_1} - g_{/a} \rangle$. For every element d_7 of the carrier of E, there exists an element d_8 of the carrier of $E \times F$ such that $\mathcal{D}[d_7, d_8]$.

Consider V being a function from the carrier of E into the carrier of $E \times F$ such that for every element d_7 of the carrier of E, $\mathcal{D}[d_7, V(d_7)]$. Reconsider $Q_1 = Q$ as a point of the real norm space of bounded linear operators from G into F. Set $Q_2 = \|Q_1\|$. Consider d_0 being a real number such that $d_0 > 0$ and for every point d_9 of $E \times F$ such that $d_9 \neq 0_{E \times F}$ and $\|d_9\| < d_0$ holds $\|d_9\|^{-1} \cdot \|R_{/d_9}\| < \frac{1}{2 \cdot (Q_2 + 1)}$. Consider d_1 being a real number such that $0 < d_1 < d_0$ and $\mathrm{Ball}(a, d_1) \times \mathrm{Ball}(g_{/a}, d_1) \subseteq \mathrm{Ball}(z, d_0)$. Consider d_2 being a real number such that $0 < d_2$ and for every point x_1 of E such that $x_1 \in \mathrm{dom}\, g$ and $\|x_1 - a\| < d_2$ holds $\|g_{/x_1} - g_{/a}\| < d_1$. Reconsider $d_3 = \min(d_1, d_2)$ as a real number. Reconsider $d_4 = \min(d_3, r_1)$ as a real number.

For every point d_7 of E such that $d_7 \neq 0_E$ and $\|d_7\| < d_4$ holds $\|R_{/V(d_7)}\| \leqslant \frac{1}{2 \cdot (Q_2 + 1)} \cdot (\|d_7\| + \|g_{/a+d_7} - g_{/a}\|)$. For every point d_7 of E such that $d_7 \neq 0_E$ and $\|d_7\| < d_4$ holds $\|R_{1/d_7}\| \leqslant \frac{1}{2} \cdot (\|d_7\| + \|g_{/a+d_7} - g_{/a}\|)$. Set $Q_3 = \|L\|$. Reconsider $d_5 = \min(r_6, d_4)$ as a real number. For every point d_7 of E such that $d_7 \neq 0_E$ and $\|d_7\| < d_5$ holds $\|g_{/a+d_7} - g_{/a}\| \leqslant (2 \cdot Q_3 + 1) \cdot \|d_7\|$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point d_7 of E such that $d_7 \neq 0_E$ and $\|d_7\| < d$ holds $\|d_7\|^{-1} \cdot \|R_{1/d_7}\| < r$ by [4, (23)], [7, (7)], [8, (18)]. □

From now on X, Y, Z denote non trivial real Banach spaces.

Now we state the propositions:

(16) Let us consider a point u of the real norm space of bounded linear operators from X into Y. Suppose u is invertible. Then there exist real numbers K, s such that

(i) $0 \leqslant K$, and

(ii) $0 < s$, and

(iii) for every point d_6 of the real norm space of bounded linear operators from X into Y such that $\|d_6\| < s$ holds $u + d_6$ is invertible and $\|\mathrm{Inv}\, u + d_6 - \mathrm{Inv}\, u - -(\mathrm{Inv}\, u) \cdot d_6 \cdot (\mathrm{Inv}\, u)\| \leqslant K \cdot (\|d_6\| \cdot \|d_6\|)$.

(17) Let us consider a partial function I from the real norm space of bounded linear operators from X into Y to the real norm space of bounded linear operators from Y into X. Suppose $\mathrm{dom}\, I = \mathrm{InvertOpers}(X, Y)$ and for every point u of the real norm space of bounded linear operators from X into Y such that $u \in \mathrm{InvertOpers}(X, Y)$ holds $I(u) = \mathrm{Inv}\, u$. Let us consider a point u of the real norm space of bounded linear operators from X into Y. Suppose $u \in \mathrm{InvertOpers}(X, Y)$. Then

(i) I is differentiable in u, and

(ii) for every point d_6 of the real norm space of bounded linear operators from X into Y, $(I'(u))(d_6) = -(\mathrm{Inv}\, u) \cdot d_6 \cdot (\mathrm{Inv}\, u)$.

Proof: Set $S =$ the real norm space of bounded linear operators from X into Y. Set $W =$ the real norm space of bounded linear operators from Y into X. Set $N = \mathrm{InvertOpers}(X, Y)$. Define $\mathcal{C}(\text{point of } S) = -(\mathrm{Inv}\, u) \cdot \$_1 \cdot (\mathrm{Inv}\, u)$. Consider L being a function from the carrier of S into the carrier of W such that for every point x of S, $L(x) = \mathcal{C}(x)$. For every elements s, t of S, $L(s + t) = L(s) + L(t)$. For every element s of S and for every real number r, $L(r \cdot s) = r \cdot L(s)$. Define $\mathcal{D}(\text{point of } S) = \mathrm{Inv}\, u + \$_1 - \mathrm{Inv}\, u - L(\$_1)$.

Consider R being a function from the carrier of S into the carrier of W such that for every point x of S, $R(x) = \mathcal{D}(x)$. For every point x of S, $R(x) = \mathrm{Inv}\, u + x - \mathrm{Inv}\, u - -(\mathrm{Inv}\, u) \cdot x \cdot (\mathrm{Inv}\, u)$. Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from S into W. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{/z}\| < r$. Reconsider $R_0 = R$ as a rest of S, W. For every point v of S such that $v \in N$ holds $I_{/v} - I_{/u} = L_0(v - u) + R_{0/v-u}$. □

(18) There exists a partial function I from the real norm space of bounded linear operators from X into Y to the real norm space of bounded linear operators from Y into X such that

(i) $\mathrm{dom}\, I = \mathrm{InvertOpers}(X, Y)$, and

(ii) $\mathrm{rng}\, I = \mathrm{InvertOpers}(Y, X)$, and

(iii) I is one-to-one and differentiable on $\mathrm{InvertOpers}(X, Y)$, and

(iv) there exists a partial function J from the real norm space of bounded linear operators from Y into X to the real norm space of bounded linear operators from X into Y such that $J = I^{-1}$ and J is one-to-one and $\operatorname{dom} J = \operatorname{InvertOpers}(Y, X)$ and $\operatorname{rng} J = \operatorname{InvertOpers}(X, Y)$ and J is differentiable on $\operatorname{InvertOpers}(Y, X)$, and

(v) for every point u of the real norm space of bounded linear operators from X into Y such that $u \in \operatorname{InvertOpers}(X, Y)$ holds $I(u) = \operatorname{Inv} u$, and

(vi) for every points u, d_6 of the real norm space of bounded linear operators from X into Y such that $u \in \operatorname{InvertOpers}(X, Y)$ holds $(I'(u))(d_6) = -(\operatorname{Inv} u) \cdot d_6 \cdot (\operatorname{Inv} u)$.

PROOF: Consider I being a partial function from the real norm space of bounded linear operators from X into Y to the real norm space of bounded linear operators from Y into X such that $\operatorname{dom} I = \operatorname{InvertOpers}(X, Y)$ and $\operatorname{rng} I = \operatorname{InvertOpers}(Y, X)$ and I is one-to-one and continuous on $\operatorname{InvertOpers}(X, Y)$ and there exists a partial function J from the real norm space of bounded linear operators from Y into X to the real norm space of bounded linear operators from X into Y such that $J = I^{-1}$ and J is one-to-one and $\operatorname{dom} J = \operatorname{InvertOpers}(Y, X)$ and $\operatorname{rng} J = \operatorname{InvertOpers}(X, Y)$ and J is continuous on $\operatorname{InvertOpers}(Y, X)$ and for every point u of the real norm space of bounded linear operators from X into Y such that $u \in \operatorname{InvertOpers}(X, Y)$ holds $I(u) = \operatorname{Inv} u$.

Consider J being a partial function from the real norm space of bounded linear operators from Y into X to the real norm space of bounded linear operators from X into Y such that $J = I^{-1}$ and J is one-to-one and $\operatorname{dom} J = \operatorname{InvertOpers}(Y, X)$ and $\operatorname{rng} J = \operatorname{InvertOpers}(X, Y)$ and J is continuous on $\operatorname{InvertOpers}(Y, X)$. For every point u of the real norm space of bounded linear operators from X into Y such that $u \in \operatorname{InvertOpers}(X, Y)$ holds I is differentiable in u. For every point v of the real norm space of bounded linear operators from Y into X such that $v \in \operatorname{InvertOpers}(Y, X)$ holds $J(v) = \operatorname{Inv} v$ by [5, (15)]. For every point v of the real norm space of bounded linear operators from Y into X such that $v \in \operatorname{InvertOpers}(Y, X)$ holds J is differentiable in v. □

(19) Let us consider real normed spaces E, G, F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point a of E, a point b of F, a point c of G, a point z of $E \times F$, a subset A of E, a subset B of F, and a partial function g from E to F. Suppose Z is open and $\operatorname{dom} f = Z$ and A is open and B is open and $A \times B \subseteq Z$ and $z = \langle a, b \rangle$ and $f(a, b) = c$ and f is differentiable in z and $\operatorname{dom} g = A$ and $\operatorname{rng} g \subseteq B$ and $a \in A$ and $g(a) = b$ and g is continuous in a and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and $\operatorname{partdiff}(f, z)$ w.r.t. 2 is invertible. Then

(i) g is differentiable in a, and

(ii) $g'(a) = -(\mathrm{Inv\,partdiff}(f, z)\ \mathrm{w.r.t.}\ 2) \cdot (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1)$.

PROOF: Consider r_2 being a real number such that $0 < r_2$ and $\mathrm{Ball}(b, r_2) \subseteq B$. Consider r_3 being a real number such that $0 < r_3$ and for every point x_1 of E such that $x_1 \in \mathrm{dom}\, g$ and $\|x_1 - a\| < r_3$ holds $\|g_{/x_1} - g_{/a}\| < r_2$. Consider r_4 being a real number such that $0 < r_4$ and $\mathrm{Ball}(a, r_4) \subseteq A$. Set $r_1 = \min(r_3, r_4)$. Set $g_1 = g \restriction \mathrm{Ball}(a, r_1)$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in \mathrm{dom}\, g_1$ and $\|x_1 - a\| < s$ holds $\|g_{1/x_1} - g_{1/a}\| < r$. For every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$.

Reconsider $Q = (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 2)^{-1}$ as a Lipschitzian linear operator from G into F. Reconsider $P = \mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1$ as a Lipschitzian linear operator from E into G. g_1 is differentiable in a and $g_1{}'(a) = -Q \cdot P$. Consider N being a neighbourhood of a such that $N \subseteq \mathrm{dom}\, g_1$ and there exists a rest R of E, F such that for every point x of E such that $x \in N$ holds $g_{1/x} - g_{1/a} = (g_1{}'(a))(x - a) + R_{/x-a}$. Consider R being a rest of E, F such that for every point x of E such that $x \in N$ holds $g_{1/x} - g_{1/a} = (g_1{}'(a))(x - a) + R_{/x-a}$. For every point x of E such that $x \in N$ holds $g_{/x} - g_{/a} = (g_1{}'(a))(x - a) + R_{/x-a}$. □

(20) Let us consider a real normed space E, non trivial real Banach spaces G, F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point c of G, a subset A of E, a subset B of F, and a partial function g from E to F. Suppose Z is open and $\mathrm{dom}\, f = Z$ and A is open and B is open and $A \times B \subseteq Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $\mathrm{dom}\, g = A$ and $\mathrm{rng}\, g \subseteq B$ and g is continuous on A and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 2$ is invertible. Then

(i) g is differentiable on A, and

(ii) $g'_{\restriction A}$ is continuous on A, and

(iii) for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -(\mathrm{Inv\,partdiff}(f, z)\ \mathrm{w.r.t.}\ 2) \cdot (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1)$.

PROOF: For every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds g is differentiable in x and $g'(x) = -(\mathrm{Inv\,partdiff}(f, z)\ \mathrm{w.r.t.}\ 2) \cdot (\mathrm{partdiff}(f, z)\ \mathrm{w.r.t.}\ 1)$. For every point x of E such that $x \in A$ holds g is differentiable in x. Consider x_2 being a partial function from E to $E \times F$ such that $\mathrm{dom}\, x_2 = A$ and for every point x of E such that $x \in A$ holds $x_2(x) = \langle x, g(x) \rangle$ and x_2 is continuous on A. Consider B being a bilinear operator from the real norm space of bounded linear operators from E into G $\times$ the real norm space of bounded linear operators from G into F into the real norm space of bounded linear operators from E into F such that B is continuous on the carrier of (the real norm space of bounded linear operators from E into G) $\times$ (the real norm space of bounded linear operators from G into F) and for every point u of the real norm space of bounded linear operators from E into G and for every point v of the real norm space of bounded linear operators from G into F, $B(u, v) = v \cdot u$.

Consider I being a partial function from the real norm space of bounded linear operators from F into G to the real norm space of bounded linear operators from G into F such that $\mathrm{dom}\, I = \mathrm{InvertOpers}(F, G)$ and $\mathrm{rng}\, I = \mathrm{InvertOpers}(G, F)$ and I is one-to-one and continuous on $\mathrm{InvertOpers}(F, G)$ and there exists a partial function J from the real norm space of bounded linear operators from G into F to the real norm space of bounded linear operators from F into G such that $J = I^{-1}$ and J is one-to-one and $\mathrm{dom}\, J = \mathrm{InvertOpers}(G, F)$ and $\mathrm{rng}\, J = \mathrm{InvertOpers}(F, G)$ and J is continuous on $\mathrm{InvertOpers}(G, F)$ and for every point u of the real norm space of bounded linear operators from F into G such that $u \in \mathrm{InvertOpers}(F, G)$ holds $I(u) = \mathrm{Inv}\, u$. For every point x of E such that $x \in A$ holds $(g'_{\restriction A})_{/x} = -B_{/\langle ((f \restriction^1 Z) \cdot x_2)(x),\, (I \cdot (f \restriction^2 Z) \cdot x_2)(x) \rangle}$.

For every point x of E such that $x \in A$ holds $x \in \mathrm{dom}((f \restriction^1 Z) \cdot x_2)$ and $(f \restriction^1 Z) \cdot x_2$ is continuous in x. For every point x of E such that $x \in A$ holds $x \in \mathrm{dom}(I \cdot (f \restriction^2 Z) \cdot x_2)$ and $I \cdot (f \restriction^2 Z) \cdot x_2$ is continuous in x. Consider H being a partial function from E to the real norm space of bounded linear operators from E into F such that $\mathrm{dom}\, H = A$ and for every point x of E such that $x \in A$ holds $H(x) = B(((f \restriction^1 Z) \cdot x_2)(x), (I \cdot (f \restriction^2 Z) \cdot x_2)(x))$ and H is continuous on A. For every point x_0 of E such that $x_0 \in A$ holds $B(\langle ((f \restriction^1 Z) \cdot x_2)(x_0),\, (I \cdot (f \restriction^2 Z) \cdot x_2)(x_0) \rangle) = B_{/\langle ((f \restriction^1 Z) \cdot x_2)(x_0),\, (I \cdot (f \restriction^2 Z) \cdot x_2)(x_0) \rangle}$. For every point x_0 of E such that $x_0 \in A$ holds $g'_{\restriction A} \restriction A$ is continuous in x_0. □

(21) Let us consider a real normed space E, non trivial real Banach spaces G, F, a subset Z of $E \times F$, a partial function f from $E \times F$ to G, a point a of E, a point b of F, a point c of G, and a point z of $E \times F$. Suppose Z is open and $\mathrm{dom}\, f = Z$ and f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z and $\langle a, b \rangle \in Z$ and $f(a, b) = c$ and $z = \langle a, b \rangle$ and $\mathrm{partdiff}(f, z)$ w.r.t. 2 is invertible. Then there exist real numbers r_1, r_2 such that

(i) $0 < r_1$, and

(ii) $0 < r_2$, and

(iii) $\mathrm{Ball}(a, r_1) \times \overline{\mathrm{Ball}}(b, r_2) \subseteq Z$, and

(iv) for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ there exists a point y of F such that $y \in \mathrm{Ball}(b, r_2)$ and $f(x, y) = c$, and

(v) for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \mathrm{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$, and

(vi) there exists a partial function g from E to F such that $\mathrm{dom}\, g = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g \subseteq \mathrm{Ball}(b, r_2)$ and g is continuous on $\mathrm{Ball}(a, r_1)$ and $g(a) = b$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and g is differentiable on $\mathrm{Ball}(a, r_1)$ and $g'_{\restriction \mathrm{Ball}(a, r_1)}$ is continuous on $\mathrm{Ball}(a, r_1)$ and for every point x of E and for every point z of $E \times F$ such that $x \in \mathrm{Ball}(a, r_1)$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -(\mathrm{Inv}\, \mathrm{partdiff}(f, z)\, \mathrm{w.r.t.}\, 2) \cdot (\mathrm{partdiff}(f, z)\, \mathrm{w.r.t.}\, 1)$ and for every point x of E and for every point z of $E \times F$ such that $x \in \mathrm{Ball}(a, r_1)$ and $z = \langle x, g(x) \rangle$ holds $\mathrm{partdiff}(f, z)\, \mathrm{w.r.t.}\, 2$ is invertible, and

(vii) for every partial functions g_1, g_2 from E to F such that $\mathrm{dom}\, g_1 = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g_1 \subseteq \mathrm{Ball}(b, r_2)$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\mathrm{dom}\, g_2 = \mathrm{Ball}(a, r_1)$ and $\mathrm{rng}\, g_2 \subseteq \mathrm{Ball}(b, r_2)$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

PROOF: Set $P = f_0 \restriction^2 Z_0$. Consider p_1 being a real number such that $0 < p_1$ and $\mathrm{Ball}(P_{/z}, p_1) \subseteq \mathrm{InvertOpers}(F, G)$. Consider s_1 being a real number such that $0 < s_1$ and for every point z_1 of $E \times F$ such that $z_1 \in Z_0$ and $\|z_1 - z\| < s_1$ holds $\|P_{/z_1} - P_{/z}\| < p_1$. Consider s_2 being a real number such that $0 < s_2$ and $\mathrm{Ball}(z, s_2) \subseteq Z_0$. Set $s = \min(s_1, s_2)$. Set $Z = \mathrm{Ball}(z, s)$. Set $f = f_0 \restriction Z$. Set $D = f'_{\restriction Z}$. For every point z of $E \times F$ such that $z \in Z$ holds ${f_0}'(z) = f'(z)$. For every point x_0 of $E \times F$ and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $E \times F$ such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|D_{/x_1} - D_{/x_0}\| < r$. For every point z of $E \times F$ such that $z \in Z$ holds $\mathrm{partdiff}(f_0, z)\, \mathrm{w.r.t.}\, 1 = \mathrm{partdiff}(f, z)\, \mathrm{w.r.t.}\, 1$ and $\mathrm{partdiff}(f_0, z)\, \mathrm{w.r.t.}\, 2 = \mathrm{partdiff}(f, z)\, \mathrm{w.r.t.}\, 2$.

Consider r_1, r_2 being real numbers such that $0 < r_1$ and $0 < r_2$ and $\mathrm{Ball}(a, r_1) \times \overline{\mathrm{Ball}}(b, r_2) \subseteq Z$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ there exists a point y of F such that $y \in \mathrm{Ball}(b, r_2)$ and $f(x, y) = c$ and for every point x of E such that $x \in \mathrm{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \mathrm{Ball}(b, r_2)$ and $f(x, y_1) = c$ and $f(x, y_2) = c$ holds $y_1 = y_2$ and there exists a partial function g from E to F such that g is continuous on $\mathrm{Ball}(a, r_1)$ and $\mathrm{dom}\, g = \mathrm{Ball}(a, r_1)$ and

$\operatorname{rng} g \subseteq \operatorname{Ball}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g(x)) = c$ and for every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g_2(x)) = c$ holds $g_1 = g_2$.

For every point x of E and for every point y of F such that $x \in \operatorname{Ball}(a, r_1)$ and $y \in \operatorname{Ball}(b, r_2)$ holds $f_0(x, y) = f(x, y)$. For every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ there exists a point y of F such that $y \in \operatorname{Ball}(b, r_2)$ and $f_0(x, y) = c$. For every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ for every points y_1, y_2 of F such that y_1, $y_2 \in \operatorname{Ball}(b, r_2)$ and $f_0(x, y_1) = c$ and $f_0(x, y_2) = c$ holds $y_1 = y_2$. Consider g being a partial function from E to F such that g is continuous on $\operatorname{Ball}(a, r_1)$ and $\operatorname{dom} g = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g \subseteq \operatorname{Ball}(b, r_2)$ and $g(a) = b$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f(x, g(x)) = c$. For every point x of E and for every point w of $E \times F$ such that $x \in \operatorname{Ball}(a, r_1)$ and $w = \langle x, g(x) \rangle$ holds $\operatorname{partdiff}(f_0, w)$ w.r.t. 2 is invertible. For every point x of E and for every point w of $E \times F$ such that $x \in \operatorname{Ball}(a, r_1)$ and $w = \langle x, g(x) \rangle$ holds $\operatorname{partdiff}(f, w)$ w.r.t. 2 is invertible.

For every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f_0(x, g(x)) = c$. g is differentiable on $\operatorname{Ball}(a, r_1)$ and $g'_{\restriction \operatorname{Ball}(a, r_1)}$ is continuous on $\operatorname{Ball}(a, r_1)$ and for every point x of E and for every point z of $E \times F$ such that $x \in \operatorname{Ball}(a, r_1)$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -(\operatorname{Inv} \operatorname{partdiff}(f_0, z)$ w.r.t. $2) \cdot (\operatorname{partdiff}(f_0, z)$ w.r.t. $1)$. For every partial functions g_1, g_2 from E to F such that $\operatorname{dom} g_1 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_1 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f_0(x, g_1(x)) = c$ and $\operatorname{dom} g_2 = \operatorname{Ball}(a, r_1)$ and $\operatorname{rng} g_2 \subseteq \operatorname{Ball}(b, r_2)$ and for every point x of E such that $x \in \operatorname{Ball}(a, r_1)$ holds $f_0(x, g_2(x)) = c$ holds $g_1 = g_2$. $\square$

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