

Some Facts about Trigonometry and Euclidean Geometry

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Summary. We calculate the values of the trigonometric functions for angles: $\frac{\pi}{3}$ and $\frac{\pi}{6}$, by [16]. After defining some trigonometric identities, we demonstrate conventional trigonometric formulas in the triangle, and the geometric property, by [14], of the triangle inscribed in a semicircle, by the proposition 3.31 in [15]. Then we define the diameter of the circumscribed circle of a triangle using the definition of the area of a triangle and prove some identities of a triangle [9]. We conclude by indicating that the diameter of a circle is twice the length of the radius.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [10], [11], [19], [25], [3], [12], [5], [21], [2], [28], [6], [7], [24], [29], [23], [18], [26], [27], [13], and [8].

1. VALUES OF THE TRIGONOMETRIC FUNCTIONS FOR ANGLES: $\frac{\pi}{3}$ AND $\frac{\pi}{6}$

Let us consider a real number a . Now we state the propositions:

- (1) $\sin(\pi - a) = \sin a$.
- (2) $\cos(\pi - a) = -\cos a$.
- (3) $\sin(2 \cdot \pi - a) = -\sin a$.
- (4) $\cos(2 \cdot \pi - a) = \cos a$.
- (5) $\sin(-2 \cdot \pi + a) = \sin a$.

(6) $\cos(-2 \cdot \pi + a) = \cos a.$

(7) $\sin\left(\frac{3\pi}{2} + a\right) = -\cos a.$

(8) $\cos\left(\frac{3\pi}{2} + a\right) = \sin a.$

(9) $\sin\left(\frac{3\pi}{2} + a\right) = -\sin\left(\frac{\pi}{2} - a\right).$ The theorem is a consequence of (7).

(10) $\cos\left(\frac{3\pi}{2} + a\right) = \cos\left(\frac{\pi}{2} - a\right).$ The theorem is a consequence of (8).

(11) $\sin\left(\frac{2\pi}{3} - a\right) = \sin\left(\frac{\pi}{3} + a\right).$

(12) $\cos\left(\frac{2\pi}{3} - a\right) = -\cos\left(\frac{\pi}{3} + a\right).$

(13) $\sin\left(\frac{2\pi}{3} + a\right) = \sin\left(\frac{\pi}{3} - a\right).$

Now we state the propositions:

(14) $\cos \frac{\pi}{3} = \frac{1}{2}.$

(15) $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$

PROOF: $\sin \frac{\pi}{3} \geq 0$ by [20, (5)], [29, (79), (81)]. \square

(16) $\operatorname{tg} \frac{\pi}{3} = \sqrt{3}.$ The theorem is a consequence of (14) and (15).

(17) $\sin \frac{\pi}{6} = \frac{1}{2}.$ The theorem is a consequence of (14).

(18) $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$ The theorem is a consequence of (15).

(19) $\operatorname{tg} \frac{\pi}{6} = \frac{\sqrt{3}}{3}.$ The theorem is a consequence of (17) and (18).

(20) (i) $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2},$ and

(ii) $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$ and

(iii) $\operatorname{tg}\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3},$ and

(iv) $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},$ and

(v) $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2},$ and

(vi) $\operatorname{tg}\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$

(21) (i) $\arcsin \frac{1}{2} = \frac{\pi}{6},$ and

(ii) $\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$

The theorem is a consequence of (15) and (17).

(22) $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$ The theorem is a consequence of (11) and (15).

(23) $\cos \frac{2\pi}{3} = -\frac{1}{2}.$ The theorem is a consequence of (12) and (14).

2. SOME TRIGONOMETRIC IDENTITIES

Now we state the proposition:

(24) Let us consider a real number x . Then $(\sin(-x))^2 = (\sin x)^2.$

Let us consider real numbers x, y, z . Now we state the propositions:

(25) If $x + y + z = \pi$, then $(\sin x)^2 + (\sin y)^2 - 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2.$

- (26) If $x - y + z = \pi$, then $(\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2$.
The theorem is a consequence of (24) and (25).
- (27) Suppose $x - (-2 \cdot \pi + y) + z = \pi$. Then $(\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2$. The theorem is a consequence of (24), (5), and (25).
- (28) If $\pi - x - (\pi - y) + z = \pi$, then $(\sin x)^2 + (\sin y)^2 + 2 \cdot \sin x \cdot \sin y \cdot \cos z = (\sin z)^2$. The theorem is a consequence of (24), (1), and (25).

Now we state the proposition:

- (29) Let us consider a real number a . Then $\sin(3 \cdot a) = 4 \cdot \sin a \cdot \sin(\frac{\pi}{3} + a) \cdot \sin(\frac{\pi}{3} - a)$. The theorem is a consequence of (15).

3. TRIGONOMETRIC FUNCTIONS AND RIGHT TRIANGLE

Let us consider points A, B, C of \mathcal{E}_1^2 .

Let us assume that A, B, C form a triangle. Now we state the propositions:

- (30) (i) $\sphericalangle(A, B, C)$ is not zero, and
 (ii) $\sphericalangle(B, C, A)$ is not zero, and
 (iii) $\sphericalangle(C, A, B)$ is not zero, and
 (iv) $\sphericalangle(A, C, B)$ is not zero, and
 (v) $\sphericalangle(C, B, A)$ is not zero, and
 (vi) $\sphericalangle(B, A, C)$ is not zero.
- (31) (i) $\sphericalangle(A, B, C) = 2 \cdot \pi - \sphericalangle(C, B, A)$, and
 (ii) $\sphericalangle(B, C, A) = 2 \cdot \pi - \sphericalangle(A, C, B)$, and
 (iii) $\sphericalangle(C, A, B) = 2 \cdot \pi - \sphericalangle(B, A, C)$, and
 (iv) $\sphericalangle(B, A, C) = 2 \cdot \pi - \sphericalangle(C, A, B)$, and
 (v) $\sphericalangle(A, C, B) = 2 \cdot \pi - \sphericalangle(B, C, A)$, and
 (vi) $\sphericalangle(C, B, A) = 2 \cdot \pi - \sphericalangle(A, B, C)$.

Now we state the proposition:

- (32) Suppose A, B, C form a triangle and $|(B - A, C - A)| = 0$. Then
 (i) $|C - B| \cdot \sin \sphericalangle(C, B, A) = |A - C|$, or
 (ii) $|C - B| \cdot (-\sin \sphericalangle(C, B, A)) = |A - C|$.

Let us assume that A, B, C form a triangle and $\sphericalangle(B, A, C) = \frac{\pi}{2}$. Now we state the propositions:

- (33) $\sphericalangle(C, B, A) + \sphericalangle(A, C, B) = \frac{\pi}{2}$.
- (34) (i) $|C - B| \cdot \sin \sphericalangle(C, B, A) = |A - C|$, and

- (ii) $|C - B| \cdot \sin \sphericalangle(A, C, B) = |A - B|$, and
 (iii) $|C - B| \cdot \cos \sphericalangle(C, B, A) = |A - B|$, and
 (iv) $|C - B| \cdot \cos \sphericalangle(A, C, B) = |A - C|$.
- (35) (i) $\operatorname{tg} \sphericalangle(A, C, B) = \frac{|A-B|}{|A-C|}$, and
 (ii) $\operatorname{tg} \sphericalangle(C, B, A) = \frac{|A-C|}{|A-B|}$.

The theorem is a consequence of (34).

4. TRIANGLE INSCRIBED IN A SEMICIRCLE IS A RIGHT TRIANGLE

Let a, b be real numbers and r be a negative real number. Let us note that $\operatorname{circle}(a, b, r)$ is empty.

Now we state the proposition:

- (36) Let us consider real numbers a, b . Then $\operatorname{circle}(a, b, 0) = \{[a, b]\}$.

Let a, b be real numbers. One can verify that $\operatorname{circle}(a, b, 0)$ is trivial.

Now we state the propositions:

- (37) Let us consider points A, B, C of \mathcal{E}_T^2 , and real numbers a, b, r . Suppose A, B, C form a triangle and $A, B \in \operatorname{circle}(a, b, r)$. Then r is positive. The theorem is a consequence of (36).

- (38) Let us consider a point A of \mathcal{E}_T^2 , real numbers a, b , and a positive real number r . If $A \in \operatorname{circle}(a, b, r)$, then $A \neq [a, b]$.

- (39) Let us consider points A, B, C of \mathcal{E}_T^2 , and real numbers a, b, r . Suppose A, B, C form a triangle and $\sphericalangle(C, B, A), \sphericalangle(B, A, C) \in]0, \pi[$ and $A, B, C \in \operatorname{circle}(a, b, r)$ and $[a, b] \in \mathcal{L}(A, C)$. Then $\sphericalangle(C, B, A) = \frac{\pi}{2}$.

PROOF: Set $O = [a, b]$. Consider J_1 being a point of \mathcal{E}_T^2 such that $A = J_1$ and $|J_1 - [a, b]| = r$. Consider J_2 being a point of \mathcal{E}_T^2 such that $B = J_2$ and $|J_2 - [a, b]| = r$. Consider J_3 being a point of \mathcal{E}_T^2 such that $C = J_3$ and $|J_3 - [a, b]| = r$. r is positive. $O \neq A$ and $O \neq C$. $\sphericalangle(C, B, O) < \pi$ by [25, (16), (9)], [19, (47)]. A, O, B form a triangle and C, O, B form a triangle by (37), (38), [6, (72), (75)]. $\sphericalangle(C, B, O) + \sphericalangle(O, C, B) + \sphericalangle(O, B, A) + \sphericalangle(B, A, O) = \pi$ or $\sphericalangle(C, B, O) + \sphericalangle(O, C, B) + \sphericalangle(O, B, A) + \sphericalangle(B, A, O) = -\pi$ by [25, (13)], [19, (47)]. $\sphericalangle(O, C, B) = \sphericalangle(C, B, O)$ and $\sphericalangle(B, A, O) = \sphericalangle(O, B, A)$. \square

- (40) Let us consider points A, B, C of \mathcal{E}_T^2 , and a positive real number r . Suppose $\sphericalangle(A, B, C)$ is not zero. Then $\sin(r \cdot \sphericalangle(C, B, A)) = \sin(r \cdot 2 \cdot \pi) \cdot \cos(r \cdot \sphericalangle(A, B, C)) - \cos(r \cdot 2 \cdot \pi) \cdot \sin(r \cdot \sphericalangle(A, B, C))$.

- (41) Let us consider points A, B, C of \mathcal{E}_T^2 . Suppose $\sphericalangle(A, B, C)$ is not zero. Then $\sin \frac{\sphericalangle(C, B, A)}{3} = \frac{\sqrt{3}}{2} \cdot \cos \frac{\sphericalangle(A, B, C)}{3} + \frac{1}{2} \cdot \sin \frac{\sphericalangle(A, B, C)}{3}$. The theorem is a consequence of (40), (22), and (23).

5. DIAMETER OF THE CIRCUMCIRCLE OF A TRIANGLE

Let us consider points A, B, C of \mathcal{E}_T^2 . Now we state the propositions:

(42) (i) area of $\Delta(A, B, C) = \text{area of } \Delta(B, C, A)$, and

(ii) area of $\Delta(A, B, C) = \text{area of } \Delta(C, A, B)$.

(43) area of $\Delta(A, B, C) = -(\text{area of } \Delta(B, A, C))$.

Let A, B, C be points of \mathcal{E}_T^2 . The functor $\varnothing_{\square}(A, B, C)$ yielding a real number is defined by the term

(Def. 1) $\frac{|A-B| \cdot |B-C| \cdot |C-A|}{2 \cdot \text{area of } \Delta(A, B, C)}$.

Let us consider points A, B, C of \mathcal{E}_T^2 .

Let us assume that A, B, C form a triangle. Now we state the propositions:

(44) $\varnothing_{\square}(A, B, C) = \frac{|C-A|}{\sin \sphericalangle(C, B, A)}$.

(45) $\varnothing_{\square}(A, B, C) = -\frac{|C-A|}{\sin \sphericalangle(A, B, C)}$. The theorem is a consequence of (44).

Now we state the proposition:

(46) $\varnothing_{\square}(A, B, C) = \varnothing_{\square}(B, C, A)$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(47) $\varnothing_{\square}(A, B, C) = -\varnothing_{\square}(B, A, C)$. The theorem is a consequence of (43).

(48) $\varnothing_{\square}(A, B, C) = -\varnothing_{\square}(A, C, B)$. The theorem is a consequence of (42) and (47).

(49) $\varnothing_{\square}(A, B, C) = -\varnothing_{\square}(C, B, A)$. The theorem is a consequence of (48) and (42).

6. SOME IDENTITIES OF A TRIANGLE

Let us consider points A, B, C of \mathcal{E}_T^2 .

Let us assume that A, B, C form a triangle. Now we state the propositions:

(50) (i) $|A - B| = \varnothing_{\square}(A, B, C) \cdot \sin \sphericalangle(A, C, B)$, and

(ii) $|B - C| = \varnothing_{\square}(A, B, C) \cdot \sin \sphericalangle(B, A, C)$, and

(iii) $|C - A| = \varnothing_{\square}(A, B, C) \cdot \sin \sphericalangle(C, B, A)$.

The theorem is a consequence of (42).

(51) $|A - B| = \varnothing_{\square}(A, B, C) \cdot 4 \cdot \sin \frac{\sphericalangle(A, C, B)}{3} \cdot \sin(\frac{\pi}{3} + \frac{\sphericalangle(A, C, B)}{3}) \cdot \sin(\frac{\pi}{3} - \frac{\sphericalangle(A, C, B)}{3})$.

The theorem is a consequence of (29).

Let us consider points A, B, C, P of \mathcal{E}_T^2 . Now we state the propositions:

(52) Suppose A, B, P are mutually different and $\sphericalangle(P, B, A) = \frac{\sphericalangle(C, B, A)}{3}$ and $\sphericalangle(B, A, P) = \frac{\sphericalangle(B, A, C)}{3}$ and $\sphericalangle(A, P, B) < \pi$. Then $|A - P| \cdot \sin(\pi - (\frac{\sphericalangle(C, B, A)}{3} + \frac{\sphericalangle(B, A, C)}{3})) = |A - B| \cdot \sin \frac{\sphericalangle(C, B, A)}{3}$.

- (53) Suppose A, B, P are mutually different and $\sphericalangle(P, B, A) = \frac{\sphericalangle(C, B, A)}{3}$ and $\sphericalangle(B, A, P) = \frac{\sphericalangle(B, A, C)}{3}$ and $\sphericalangle(A, P, B) < \pi$ and $\frac{\sphericalangle(C, B, A)}{3} + \frac{\sphericalangle(B, A, C)}{3} + \frac{\sphericalangle(A, C, B)}{3} = \frac{\pi}{3}$. Then $|A - P| \cdot \sin(\frac{2\pi}{3} + \frac{\sphericalangle(A, C, B)}{3}) = |A - B| \cdot \sin \frac{\sphericalangle(C, B, A)}{3}$.

Now we state the proposition:

- (54) Let us consider points A, B, C of \mathcal{E}_T^2 . Suppose A, B, C form a triangle and $\sphericalangle(C, A, B) < \pi$. Then
- (i) $\sphericalangle(C, B, A) + \sphericalangle(B, A, C) + \sphericalangle(A, C, B) = 5 \cdot \pi$, and
 - (ii) $\sphericalangle(C, A, B) + \sphericalangle(A, B, C) + \sphericalangle(B, C, A) = \pi$.

Let us consider points A, B, C, P of \mathcal{E}_T^2 . Now we state the propositions:

- (55) Suppose A, B, C form a triangle and $\sphericalangle(C, B, A) < \pi$ and A, B, P are mutually different and $\sphericalangle(P, B, A) = \frac{\sphericalangle(C, B, A)}{3}$ and $\sphericalangle(B, A, P) = \frac{\sphericalangle(B, A, C)}{3}$ and $\sphericalangle(A, P, B) < \pi$. Then $|A - P| \cdot \sin(\frac{\pi}{3} - \frac{\sphericalangle(A, C, B)}{3}) = |A - B| \cdot \sin \frac{\sphericalangle(C, B, A)}{3}$. The theorem is a consequence of (1).
- (56) Suppose A, B, C form a triangle and A, B, P form a triangle and $\sphericalangle(C, B, A) < \pi$ and $\sphericalangle(A, P, B) < \pi$ and $\sphericalangle(P, B, A) = \frac{\sphericalangle(C, B, A)}{3}$ and $\sphericalangle(B, A, P) = \frac{\sphericalangle(B, A, C)}{3}$ and $\sin(\frac{\pi}{3} - \frac{\sphericalangle(A, C, B)}{3}) \neq 0$. Then $|A - P| = -\varnothing_{\square}(C, B, A) \cdot 4 \cdot \sin \frac{\sphericalangle(A, C, B)}{3} \cdot \sin(\frac{\pi}{3} + \frac{\sphericalangle(A, C, B)}{3}) \cdot \sin \frac{\sphericalangle(C, B, A)}{3}$. The theorem is a consequence of (53), (29), (50), (13), and (49).

7. DIAMETER OF A CIRCLE

Now we state the propositions:

- (57) Let us consider points A, B, C of \mathcal{E}_T^2 . Suppose A, B, C are mutually different and $C \in \mathcal{L}(A, B)$. Then $|A - B| = |A - C| + |C - B|$.
- (58) Let us consider points A, B of \mathcal{E}_T^2 , real numbers a, b , and a positive real number r . Suppose $A, B, [a, b]$ are mutually different and $A, B \in \text{circle}(a, b, r)$ and $[a, b] \in \mathcal{L}(A, B)$. Then $|A - B| = 2 \cdot r$. The theorem is a consequence of (57).
- (59) Let us consider real numbers a, b , a positive real number r , and a subset C of \mathcal{E}^2 . If $C = \text{circle}(a, b, r)$, then $\varnothing C = 2 \cdot r$.

PROOF: For every points x, y of \mathcal{E}^2 such that $x, y \in C$ holds $\rho(x, y) \leq 2 \cdot r$ by [11, (22), (67)], [17, (4)], [22, (5)]. For every real number s such that for every points x, y of \mathcal{E}^2 such that $x, y \in C$ holds $\rho(x, y) \leq s$ holds $2 \cdot r \leq s$ by [11, (62)], [4, (12)], [19, (24)], [26, (22)]. \square

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