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Differential Equations on Functions from ℝ into Real Banach Space[1]

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Summary. In this article, we describe the differential equations on functions from ℝ into real Banach space. The descriptions are based on the article [20]. As preliminary to the proof of these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [32]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [30].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [34], [31], [33], [1], [15], [25], [32], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [37], [38], [9], [35], [36], [17], and [10].

1. Some Properties of Differentiable Functions on Real Normed Space

From now on Y denotes a real normed space.

Now we state the propositions:

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(1) Let us consider a real normed space Y, a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

  (i) $J = \mathrm{proj}(1, 1)$, and

  (ii) $x_0 \in \mathrm{dom}\, f$, and

  (iii) $y_0 \in \mathrm{dom}\, g$, and

  (iv) $x_0 = \langle y_0 \rangle$, and

  (v) $f = g \cdot J$.

Then f is continuous in x_0 if and only if g is continuous in y_0. PROOF: If f is continuous in x_0, then g is continuous in y_0 by [14, (2)], [6, (39)], [37, (36)]. □

(2) Let us consider a real normed space Y, a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

  (i) $I = (\mathrm{proj}(1, 1)$ **qua** function$)^{-1}$, and

  (ii) $x_0 \in \mathrm{dom}\, f$, and

  (iii) $y_0 \in \mathrm{dom}\, g$, and

  (iv) $x_0 = \langle y_0 \rangle$, and

  (v) $f \cdot I = g$.

Then f is continuous in x_0 if and only if g is continuous in y_0. PROOF: If f is continuous in x_0, then g is continuous in y_0 by [14, (1)], [21, (33)], [26, (15)]. □

(3) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Suppose $I = (\mathrm{proj}(1, 1)$ **qua** function$)^{-1}$. Then

  (i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, $R \cdot I$ is a rest of Y, and

  (ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y, $L \cdot I$ is a linear of Y.

PROOF: For every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, $R \cdot I$ is a rest of Y by [15, (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from $\mathcal{R}^1$ into Y. Reconsider $L_1 = L_0 \cdot I$ as a partial function from $\mathbb{R}$ to Y. Reconsider $r = L_1(jj)$ as a point of Y. For every real number p, $L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. □

(4) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$. Suppose $J = \mathrm{proj}(1, 1)$. Then

  (i) for every rest R of Y, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y, and

  (ii) for every linear L of Y, $L \cdot J$ is a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y.

PROOF: For every rest R of Y, $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y by [14, (4)], [15, (6)], [5, (47)]. Consider r being a point of Y such that for every real number p, $L_p = p \cdot r$. □

(5) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to Y. Suppose

(i) $I = (\mathrm{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) f is differentiable in x_0.

Then

(vii) g is differentiable in y_0, and

(viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and

(ix) for every element r of $\mathbb{R}$, $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). PROOF: Consider N_1 being a neighbourhood of x_0 such that $N_1 \subseteq \mathrm{dom}\, f$ and there exists a point L of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into Y and there exists a rest R of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y such that for every point x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$. Consider e being a real number such that $0 < e$ and $\{z$, where z is a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e\} \subseteq N_1$. Consider L being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into Y, R being a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, Y such that for every point x_3 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3 - x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of Y. Reconsider $L_0 = L \cdot I$ as a linear of Y. Set $N = \{z$, where z is a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e\}$. $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $N_0 = \{z$, where z is an element of $\mathbb{R} : |z - y_0| < e\}$. $]y_0 - e, y_0 + e[\subseteq N_0$ by [28, (1)]. $N_0 \subseteq]y_0 - e, y_0 + e[$ by [28, (1)]. For every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{0 y_1 - y_0} + R_{0 y_1 - y_0}$ by [6, (12)], [7, (35)], [14, (3)]. □

(6) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a real number y_0, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to Y. Suppose

(i) $I = (\mathrm{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0. The theorem is a consequence of (5) and (4). PROOF: Reconsider $J = \mathrm{proj}(1,1)$ as a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$. Consider N_0 being a neighbourhood of y_0 such that $N_0 \subseteq \mathrm{dom}(f \cdot I)$ and there exists a linear L of Y and there exists a rest R of Y such that for every real number y such that $y \in N_0$ holds $(f \cdot I)_y - (f \cdot I)_{y_0} = L_{y-y_0} + R_{y-y_0}$. Consider e_0 being a real number such that $0 < e_0$ and $N_0 =]y_0 - e_0, y_0 + e_0[$. Reconsider $e = e_0$ as an element of $\mathbb{R}$. Set $N = \{z,$ where z is a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle : \|z - x_0\| < e\}$. Consider L being a linear of Y, R being a rest of Y such that for every real number y_1 such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1 - y_0} + R_{y_1 - y_0}$. Reconsider $R_0 = R \cdot J$ as a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, Y. Reconsider $L_0 = L \cdot J$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y. $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. For every point y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $y \in N$ holds $f_y - f_{x_0} = L_0(y - x_0) + R_{0\,y-x_0}$ by [6, (13)], [7, (35)], [14, (4)]. □

(7) Let us consider a function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $J = \mathrm{proj}(1,1)$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f = g \cdot J$.

Then f is differentiable in x_0 if and only if g is differentiable in y_0. The theorem is a consequence of (6).

(8) Let us consider a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, an element y_0 of $\mathbb{R}$, a partial function g from $\mathbb{R}$ to Y, and a partial function f from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to Y. Suppose

(i) $I = (\mathrm{proj}(1,1)$ **qua** function$)^{-1}$, and

(ii) $x_0 \in \mathrm{dom}\, f$, and

(iii) $y_0 \in \mathrm{dom}\, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) f is differentiable in x_0.

Then $\|g'(y_0)\| = \|f'(x_0)\|$. The theorem is a consequence of (5). PROOF: Reconsider $d_1 = f'(x_0)$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into Y. Set $A = \mathrm{PreNorms}(d_1)$. For every real number r such that $r \in A$ holds $r \leqslant \|g'(y_0)\|$ by [14, (1), (4)]. □

Let us consider real numbers a, b, z and points p, q, x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Now we state the propositions:

(9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in]a, b[$, then $x \in]p, q[$, and

(ii) if $x \in]p, q[$, then $a \neq b$ and if $a < b$, then $z \in]a, b[$ and if $a > b$, then $z \in]b, a[$.

(10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in [a, b]$, then $x \in [p, q]$, and

(ii) if $x \in [p, q]$, then if $a \leqslant b$, then $z \in [a, b]$ and if $a \geqslant b$, then $z \in [b, a]$.

Now we state the propositions:

(11) Let us consider real numbers a, b, points p, q of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and a function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Suppose

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $I = (\text{proj}(1, 1)$ **qua** function$)^{-1}$.

Then

(iv) if $a \leqslant b$, then $I^\circ[a, b] = [p, q]$, and

(v) if $a < b$, then $I^\circ]a, b[=]p, q[$.

The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space Y, a partial function g from $\mathbb{R}$ to the carrier of Y, and real numbers a, b, M. Suppose

(i) $a \leqslant b$, and

(ii) $[a, b] \subseteq \text{dom}\, g$, and

(iii) for every real number x such that $x \in [a, b]$ holds g is continuous in x, and

(iv) for every real number x such that $x \in]a, b[$ holds g is differentiable in x, and

(v) for every real number x such that $x \in]a, b[$ holds $\|g'(x)\| \leqslant M$.

Then $\|g_b - g_a\| \leqslant M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).

2. Differential Equations

In the sequel X, Y denote real Banach spaces, Z denotes an open subset of $\mathbb{R}$, a, b, c, d, e, r, x_0 denote real numbers, y_0 denotes a vector of X, and G denotes a function from X into X.

Now we state the propositions:

(13) Let us consider a real Banach space X, a partial function F from $\mathbb{R}$ to the carrier of X, and a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $[a, b] \subseteq \operatorname{dom} f$, and

(ii) $]a, b[\subseteq \operatorname{dom} F$, and

(iii) for every real number x such that $x \in]a, b[$ holds $F_x = \int\limits_a^x f(x)dx$, and

(iv) $x_0 \in]a, b[$, and

(v) f is continuous in x_0.

Then

(vi) F is differentiable in x_0, and

(vii) $F'(x_0) = f_{x_0}$.

(14) Let us consider a partial function F from $\mathbb{R}$ to the carrier of X and a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $\operatorname{dom} f = [a, b]$, and

(ii) $\operatorname{dom} F = [a, b]$, and

(iii) for every real number t such that $t \in [a, b]$ holds $F_t = \int\limits_a^t f(x)dx$.

Let us consider a real number x. If $x \in [a, b]$, then F is continuous in x.

(15) Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X. If $a \in \operatorname{dom} f$, then $\int\limits_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X and a partial function g from $\mathbb{R}$ to the carrier of X. Now we state the propositions:

(16) Suppose $a \leqslant b$ and $\operatorname{dom} f = [a, b]$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t f(x)dx$. Then $g_a = y_0$.

(17) Suppose $\operatorname{dom} f = [a,b]$ and $\operatorname{dom} g = [a,b]$ and $Z =]a,b[$ and for every real number t such that $t \in [a,b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then

(i) g is continuous and differentiable on Z, and

(ii) for every real number t such that $t \in Z$ holds $g'(t) = f_t$.

Let us consider a partial function f from $\mathbb{R}$ to the carrier of X. Now we state the propositions:

(18) Suppose $a \leqslant b$ and $[a,b] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in [a,b]$ holds f is continuous in x and f is differentiable on $]a,b[$ and for every real number x such that $x \in]a,b[$ holds $f'(x) = 0_X$. Then $f_b = f_a$.

(19) Suppose $[a,b] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in [a,b]$ holds f is continuous in x and f is differentiable on $]a,b[$ and for every real number x such that $x \in]a,b[$ holds $f'(x) = 0_X$. Then $f{\restriction}]a,b[$ is constant.

Now we state the propositions:

(20) Let us consider a continuous partial function f from $\mathbb{R}$ to the carrier of X. Suppose

(i) $[a,b] = \operatorname{dom} f$, and

(ii) $f{\restriction}]a,b[$ is constant.

Let us consider a real number x. If $x \in [a,b]$, then $f_x = f_a$.

(21) Let us consider continuous partial functions y, G_1 from $\mathbb{R}$ to the carrier of X and a partial function g from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a \leqslant b$, and

(ii) $Z =]a,b[$, and

(iii) $\operatorname{dom} y = [a,b]$, and

(iv) $\operatorname{dom} g = [a,b]$, and

(v) $\operatorname{dom} G_1 = [a,b]$, and

(vi) y is differentiable on Z, and

(vii) $y_a = y_0$, and

(viii) for every real number t such that $t \in Z$ holds $y'(t) = {G_1}_t$, and

(ix) for every real number t such that $t \in [a,b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Then $y = g$. The theorem is a consequence of (17), (16), (19), and (20).
PROOF: Reconsider $h = y - g$ as a continuous partial function from $\mathbb{R}$ to the carrier of X. For every real number x such that $x \in \operatorname{dom} h$ holds $h_x = 0_X$ by [35, (15)]. For every element x of $\mathbb{R}$ such that $x \in \operatorname{dom} y$ holds $y(x) = g(x)$ by [35, (21)]. □

Let X be a real Banach space, y_0 be a vector of X, G be a function from X into X, and a, b be real numbers. Assume $a \leqslant b$ and G is continuous on $\mathrm{dom}\, G$. The functor Fredholm(G, a, b, y_0) yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X into the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X is defined by

(Def. 1) Let us consider a vector x of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X. Then there exist continuous partial functions f, g, G_1 from $\mathbb{R}$ to the carrier of X such that

(i) $x = f$, and

(ii) $it(x) = g$, and

(iii) $\mathrm{dom}\, f = [a, b]$, and

(iv) $\mathrm{dom}\, g = [a, b]$, and

(v) $G_1 = G \cdot f$, and

(vi) for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leqslant b$ and $0 < r$ and for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X and continuous partial functions g, h from $\mathbb{R}$ to the carrier of X. Suppose

(i) $g = (\mathrm{Fredholm}(G, a, b, y_0))(u)$, and

(ii) $h = (\mathrm{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number t. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leqslant (r \cdot (t - a)) \cdot \|u - v\|$. PROOF: Set $F = \mathrm{Fredholm}(G, a, b, y_0)$. Consider f_1, g_1, G_3 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $u = f_1$ and $F(u) = g_1$ and $\mathrm{dom}\, f_1 = [a, b]$ and $\mathrm{dom}\, g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number t such that $t \in [a, b]$ holds $g_{1t} = y_0 + \int\limits_a^t G_3(x)dx$. Consider f_2, g_2, G_5 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $v = f_2$ and $F(v) = g_2$ and $\mathrm{dom}\, f_2 = [a, b]$ and $\mathrm{dom}\, g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real number t such that $t \in [a, b]$ holds $g_{2t} = y_0 + \int\limits_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$. For every real number x such that $x \in [a, t]$ holds $\|G_{4x}\| \leqslant r \cdot \|u - v\|$ by [20, (26)], [6, (12)]. □

(23) Suppose $a \leqslant b$ and $0 < r$ and for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$. Let us consider vectors u, v of the $\mathbb{R}$-norm space of

continuous functions of $[a, b]$ and X, an element m of $\mathbb{N}$, and continuous partial functions g, h from $\mathbb{R}$ to the carrier of X. Suppose

(i) $g = (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and

(ii) $h = (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(v)$.

Let us consider a real number t. Suppose $t \in [a, b]$. Then $\|g_t - h_t\| \leqslant \frac{(r\cdot(t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (22). Proof: Set $F = \mathrm{Fredholm}(G, a, b, y_0)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every continuous partial functions g, h from $\mathbb{R}$ to the carrier of X such that $g = F^{\$_1+1}(u_1)$ and $h = F^{\$_1+1}(v_1)$ for every real number t such that $t \in [a, b]$ holds $\|g_t - h_t\| \leqslant \frac{(r\cdot(t-a))^{\$_1+1}}{(\$_1+1)!} \cdot \|u_1 - v_1\|$. $\mathcal{P}[0]$ by [4, (70)], [18, (5), (13)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (71)], [6, (13)], [37, (27)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(24) Let us consider a natural number m. Suppose

(i) $a \leqslant b$, and

(ii) $0 < r$, and

(iii) for every vectors y_1, y_2 of X, $\|G_{y_1} - G_{y_2}\| \leqslant r \cdot \|y_1 - y_2\|$.

Let us consider vectors u, v of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and X.

Then $\|(\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\mathrm{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leqslant \frac{(r\cdot(b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$. The theorem is a consequence of (23).

(25) If $a < b$ and G is Lipschitzian on the carrier of X, then there exists a natural number m such that $(\mathrm{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction. The theorem is a consequence of (24).

(26) If $a < b$ and G is Lipschitzian on the carrier of X, then $\mathrm{Fredholm}(G, a, b, y_0)$ has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions f, g from $\mathbb{R}$ to the carrier of X. Suppose

(i) $\mathrm{dom}\, f = [a, b]$, and

(ii) $\mathrm{dom}\, g = [a, b]$, and

(iii) $Z =]a, b[$, and

(iv) $a < b$, and

(v) G is Lipschitzian on the carrier of X, and

(vi) $g = (\mathrm{Fredholm}(G, a, b, y_0))(f)$.

Then

(vii) $g_a = y_0$, and

(viii) g is differentiable on Z, and

(ix) for every real number t such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function y from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a < b$, and

(ii) $Z =]a, b[$, and

(iii) G is Lipschitzian on the carrier of X, and

(iv) $\operatorname{dom} y = [a, b]$, and

(v) y is differentiable on Z, and

(vi) $y_a = y_0$, and

(vii) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

Then y is a fixpoint of Fredholm(G, a, b, y_0). The theorem is a consequence of (21). PROOF: Consider f, g, G_1 being continuous partial functions from $\mathbb{R}$ to the carrier of X such that $y = f$ and $(\text{Fredholm}(G, a, b, y_0))(y) = g$ and $\operatorname{dom} f = [a, b]$ and $\operatorname{dom} g = [a, b]$ and $G_1 = G \cdot f$ and for every real number t such that $t \in [a, b]$ holds $g_t = y_0 + \int\limits_a^t G_1(x)dx$. For every real number t such that $t \in Z$ holds $y'(t) = G_{1t}$ by [6, (13)]. □

(29) Let us consider continuous partial functions y_1, y_2 from $\mathbb{R}$ to the carrier of X. Suppose

(i) $a < b$, and

(ii) $Z =]a, b[$, and

(iii) G is Lipschitzian on the carrier of X, and

(iv) $\operatorname{dom} y_1 = [a, b]$, and

(v) y_1 is differentiable on Z, and

(vi) $y_{1a} = y_0$, and

(vii) for every real number t such that $t \in Z$ holds $y_1{}'(t) = G(y_{1t})$, and

(viii) $\operatorname{dom} y_2 = [a, b]$, and

(ix) y_2 is differentiable on Z, and

(x) $y_{2a} = y_0$, and

(xi) for every real number t such that $t \in Z$ holds $y_2{}'(t) = G(y_{2t})$.

Then $y_1 = y_2$. The theorem is a consequence of (26) and (28).

(30) Suppose $a < b$ and $Z =]a, b[$ and G is Lipschitzian on the carrier of X. Then there exists a continuous partial function y from $\mathbb{R}$ to the carrier of X such that

(i) $\operatorname{dom} y = [a, b]$, and

(ii) y is differentiable on Z, and

(iii) $y_a = y_0$, and

(iv) for every real number t such that $t \in Z$ holds $y'(t) = G(y_t)$.

The theorem is a consequence of (26) and (27).

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