FORMALIZED MATHEMATICS
Vol. 20, No. 3, Pages 193–197, 2012
DOI: 10.2478/v10037-012-0022-0

VERSITA
versita.com/fm/

Transition of Consistency and Satisfiability under Language Extensions[1]

Julian J. Schlöder
Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60
D-53113 Bonn, Germany

Peter Koepke
Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
Endenicher Allee 60
D-53113 Bonn, Germany

Summary. This article is the first in a series of two Mizar articles constituting a formal proof of the Gödel Completeness theorem [17] for uncountably large languages. We follow the proof given in [18]. The present article contains the techniques required to expand formal languages. We prove that consistent or satisfiable theories retain these properties under changes to the language they are formulated in.

MML identifier: `QC_TRANS`, version: `7.14.01 4.183.1153`

The notation and terminology used in this paper have been introduced in the following papers: [8], [1], [2], [11], [16], [4], [15], [12], [13], [7], [6], [22], [3], [19], [23], [24], [5], [20], [9], [10], [21], and [14].

[1]This article is part of the first author's Bachelor thesis under the supervision of the second author.

© 2012 University of Białystok
CC-BY-SA License ver. 3.0 or later
ISSN 1426–2630(p), 1898-9934(e)

Brought to you by | Biblioteka Uniwersytecka w Bialymstoku
Authenticated
Download Date | 12/8/15 1:19 PM

1. Language Extensions

For simplicity, we adopt the following rules: A_1 denotes an alphabet, P_1 denotes a consistent subset of CQC-WFF A_1, p, r denote elements of CQC-WFF A_1, A denotes a non empty set, J denotes an interpretation of A_1 and A, v denotes an element of the valuations in A_1 and A, k denotes a natural number, l denotes a CQC-variable list of k and A_1, P denotes a predicate symbol of k and A_1, and x, y denote bound variables of A_1.

Let us consider A_1 and let A_2 be an alphabet. We say that A_2 is A_1-expanding if and only if:

(Def. 1) $A_1 \subseteq A_2$.

Let us consider A_1. Note that there exists an alphabet which is A_1-expanding.

Let A_3, A_4 be countable alphabets. One can check that there exists an alphabet which is countable, A_3-expanding, and A_4-expanding.

Let A_1, A_4 be alphabets and let P be a subset of CQC-WFF A_1. We say that P is A_4-consistent if and only if:

(Def. 2) For every subset S of CQC-WFF A_4 such that $P = S$ holds S is consistent.

Let us consider A_1. One can check that there exists a subset of CQC-WFF A_1 which is non empty and consistent.

Let us consider A_1. One can check that every subset of CQC-WFF A_1 which is consistent is also A_1-consistent and every subset of CQC-WFF A_1 which is A_1-consistent is also consistent.

For simplicity, we follow the rules: A_4 is an A_1-expanding alphabet, J_2 is an interpretation of A_4 and A, J_1 is an interpretation of A_1 and A, v_2 is an element of the valuations in A_4 and A, and v_1 is an element of the valuations in A_1 and A.

Next we state several propositions:

(1) $\mathrm{Arity}(P) = \mathrm{len}\, l$.

(2) $\mathrm{Symb}\, A_1 \subseteq \mathrm{Symb}\, A_4$.

(3) The predicate symbols of $A_1 \subseteq$ the predicate symbols of A_4.

(4) The bound variables of $A_1 \subseteq$ the bound variables of A_4.

(5) For every k holds every l is a CQC-variable list of k and A_4.

(6) P is a predicate symbol of k and A_4.

(7) For every A_1-expanding alphabet A_4 holds every p is an element of CQC-WFF A_4.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, and let p be an element of CQC-WFF A_1. The functor A_4 -Cast p yields an element of CQC-WFF A_4 and is defined by:

(Def. 3) $A_4\text{-Cast}\, p = p$.

Brought to you by | Biblioteka Uniwersytecka w Bialymstoku
Authenticated
Download Date | 12/8/15 1:19 PM

Let us consider A_1, let A_4 be an A_1-expanding alphabet, and let x be a bound variable of A_1. The functor A_4-Cast x yields a bound variable of A_4 and is defined as follows:

(Def. 4) A_4-Cast $x = x$.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, let us consider k, and let P be a predicate symbol of k and A_1. The functor A_4-Cast P yielding a predicate symbol of k and A_4 is defined as follows:

(Def. 5) A_4-Cast $P = P$.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, let us consider k, and let l be a CQC-variable list of k and A_1. The functor A_4-Cast l yielding a CQC-variable list of k and A_4 is defined as follows:

(Def. 6) A_4-Cast $l = l$.

Next we state the proposition

(8) Let given p, r, x, P, l and A_4 be an A_1-expanding alphabet. Then A_4-Cast VERUM A_1 = VERUM A_4 and A_4-Cast $P[l] = (A_4\text{-Cast } P)[A_4\text{-Cast } l]$ and A_4-Cast $\neg p = \neg(A_4\text{-Cast } p)$ and $A_4\text{-Cast}(p \wedge r) = (A_4\text{-Cast } p) \wedge (A_4\text{-Cast } r)$ and $A_4\text{-Cast } \forall_{x} p = \forall_{A_4\text{-Cast } x}(A_4\text{-Cast } p)$.

2. Downward Transfer of Consistency and Satisfiability

The following propositions are true:

(9) Suppose $J_1 = J_2 \restriction$the predicate symbols of A_1 and $v_1 = v_2 \restriction$the bound variables of A_1. Then $J_2 \models_{v_2} A_4\text{-Cast } r$ if and only if $J_1 \models_{v_1} r$.

(10) Let A_4 be an A_1-expanding alphabet and T_1 be a subset of CQC-WFF A_4. Suppose $P_1 \subseteq T_1$. Let A_2 be a non empty set, J_2 be an interpretation of A_4 and A_2, and v_2 be an element of the valuations in A_4 and A_2. If $J_2 \models_{v_2} T_1$, then there exist A, J, v such that $J \models_v P_1$.

(11) Let f be a finite sequence of elements of CQC-WFF A_4 and g be a finite sequence of elements of CQC-WFF A_1. If $f = g$, then $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$.

(12) For every p holds the still not bound in p = the still not bound in A_4-Cast p.

(13) Let p_2 be an element of CQC-WFF A_4, S be a substitution of A_1, S_2 be a substitution of A_4, x_2 be a bound variable of A_4, and given x, p. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{RestrictSub}(x, p, S) = \mathrm{RestrictSub}(x_2, p_2, S_2)$.

(14) Let p_2 be an element of CQC-WFF A_4, S be a finite substitution of A_1, S_2 be a finite substitution of A_4, and given p. If $S = S_2$ and $p = p_2$, then $\mathrm{upVar}(S, p) = \mathrm{upVar}(S_2, p_2)$.

Brought to you by | Biblioteka Uniwersytecka w Bialymstoku
Authenticated
Download Date | 12/8/15 1:19 PM

(15) Let p_2 be an element of CQC-WFF A_4, S be a substitution of A_1, S_2 be a substitution of A_4, x_2 be a bound variable of A_4, and given x, p. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{ExpandSub}(x, p, \mathrm{RestrictSub}(x, \forall_x p, S)) = \mathrm{ExpandSub}(x_2, p_2, \mathrm{RestrictSub}(x_2, \forall_{x_2} p_2, S_2))$.

(16) Let Z be an element of CQC-Sub-WFF A_1 and Z_2 be an element of CQC-Sub-WFF A_4. Suppose $Z_{\mathbf{1}}$ is universal and $(Z_2)_{\mathbf{1}}$ is universal and $\mathrm{Bound}(Z_{\mathbf{1}}) = \mathrm{Bound}((Z_2)_{\mathbf{1}})$ and $\mathrm{Scope}(Z_{\mathbf{1}}) = \mathrm{Scope}((Z_2)_{\mathbf{1}})$ and $Z = Z_2$. Then $\text{S-Bound}(^{@} Z) = \text{S-Bound}(^{@} Z_2)$.

(17) Let p_2 be an element of CQC-WFF A_4, x_2, y_2 be bound variables of A_4, and given p, x, y. If $p = p_2$ and $x = x_2$ and $y = y_2$, then $p(x, y) = p_2(x_2, y_2)$.

(18) For every consistent subset P_1 of CQC-WFF A_4 such that P_1 is a subset of CQC-WFF A_1 holds P_1 is A_1-consistent.

3. Upward Transfer of Consistency and Satisfiability

Next we state two propositions:

(19) For every p there exists a countable alphabet A_3 such that p is an element of CQC-WFF A_3 and A_1 is A_3-expanding.

(20) Let P_1 be a finite subset of CQC-WFF A_1. Then there exists a countable alphabet A_3 such that P_1 is a finite subset of CQC-WFF A_3 and A_1 is A_3-expanding.

Let us consider A_1 and let P_1 be a finite subset of CQC-WFF A_1. Note that the still not bound in P_1 is finite.

Next we state three propositions:

(21) Let T_1 be a subset of CQC-WFF A_4. Suppose $P_1 = T_1$. Let given A, J, v. Suppose $J \models_v P_1$. Then there exists a non empty set A_2 and there exists an interpretation J_2 of A_4 and A_2 and there exists an element v_2 of the valuations in A_4 and A_2 such that $J_2 \models_{v_2} T_1$.

(22) For every subset C_1 of CQC-WFF A_1 such that $C_1 \subseteq P_1$ holds C_1 is consistent.

(23) P_1 is A_4-consistent.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Patrick Braselmann and Peter Koepke. Coincidence lemma and substitution lemma. *Formalized Mathematics*, 13(**1**):17–26, 2005.

Brought to you by | Biblioteka Uniwersytecka w Bialymstoku
Authenticated
Download Date | 12/8/15 1:19 PM

[6] Patrick Braselmann and Peter Koepke. Equivalences of inconsistency and Henkin models. *Formalized Mathematics*, 13(**1**):45–48, 2005.
[7] Patrick Braselmann and Peter Koepke. Gödel's completeness theorem. *Formalized Mathematics*, 13(**1**):49–53, 2005.
[8] Patrick Braselmann and Peter Koepke. A sequent calculus for first-order logic. *Formalized Mathematics*, 13(**1**):33–39, 2005.
[9] Patrick Braselmann and Peter Koepke. Substitution in first-order formulas: Elementary properties. *Formalized Mathematics*, 13(**1**):5–15, 2005.
[10] Patrick Braselmann and Peter Koepke. Substitution in first-order formulas. Part II. The construction of first-order formulas. *Formalized Mathematics*, 13(**1**):27–32, 2005.
[11] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[12] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[13] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[14] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[15] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[16] Agata Darmochwał. A first–order predicate calculus. *Formalized Mathematics*, 1(**4**):689–695, 1990.
[17] Kurt Gödel. *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*. Monatshefte für Mathematik und Physik 37, 1930.
[18] W. Thomas H.-D. Ebbinghaus, J. Flum. *Einführung in die Mathematische Logik*. Springer-Verlag, Berlin Heidelberg, 2007.
[19] Piotr Rudnicki and Andrzej Trybulec. A first order language. *Formalized Mathematics*, 1(**2**):303–311, 1990.
[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Edmund Woronowicz. Interpretation and satisfiability in the first order logic. *Formalized Mathematics*, 1(**4**):739–743, 1990.
[22] Edmund Woronowicz. Many argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received May 7, 2012

Brought to you by | Biblioteka Uniwersytecka w Bialymstoku
Authenticated
Download Date | 12/8/15 1:19 PM