

# Simple Graphs as Simplicial Complexes: the Mycielskian of a Graph<sup>1</sup>

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**Summary.** Harary [10, p. 7] claims that Veblen [20, p. 2] first suggested to formalize simple graphs using simplicial complexes. We have developed basic terminology for simple graphs as at most 1-dimensional complexes.

We formalize this new setting and then reprove Mycielski's [12] construction resulting in a triangle-free graph with arbitrarily large chromatic number. A different formalization of similar material is in [15].

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The papers [5], [1], [4], [16], [14], [6], [9], [18], [7], [15], [2], [11], [3], [17], [13], [19], and [8] provide the terminology and notation for this paper.

## 1. PRELIMINARIES

One can prove the following propositions:

- (1) For all sets  $x, X$  holds  $\langle x, X \rangle \notin X$ .
- (2) For all sets  $x, X$  holds  $\langle x, X \rangle \neq X$ .
- (3) For all sets  $x, X$  holds  $\langle x, X \rangle \neq x$ .
- (4) For all sets  $x_1, y_1, x_2, y_2, X$  such that  $x_1, x_2 \in X$  and  $\{x_1, \langle y_1, X \rangle\} = \{x_2, \langle y_2, X \rangle\}$  holds  $x_1 = x_2$  and  $y_1 = y_2$ .
- (5) For all sets  $X, v$  such that  $3 \subseteq \overline{X}$  there exist sets  $v_1, v_2$  such that  $v_1, v_2 \in X$  and  $v_1 \neq v$  and  $v_2 \neq v$  and  $v_1 \neq v_2$ .
- (6) For every set  $x$  holds  $S_{\{x\}} = \{\{x\}\}$ .

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Let us observe that there exists a finite sequence which is finite-yielding.

The following proposition is true

- (7) Let  $X$  be a non empty finite set and  $P$  be a partition of  $X$ . If  $\overline{\overline{P}} < \overline{\overline{X}}$ , then there exist sets  $p, x, y$  such that  $p \in P$  and  $x, y \in p$  and  $x \neq y$ .

Let us note that  $\bigcup\{\emptyset\}$  is empty.

Next we state three propositions:

- (8) For every set  $x$  holds  $\bigcup\{\emptyset, \{x\}\} = \{x\}$ .
- (9) For every set  $X$  and for every subset  $s$  of  $X$  such that  $s$  is 1-element there exists a set  $x$  such that  $x \in X$  and  $s = \{x\}$ .
- (10) For every set  $X$  holds  $\overline{\overline{\{\{X, \langle x, X \rangle\}; x \text{ ranges over elements of } X: x \in X\}}} = \overline{\overline{X}}$ .

Let  $G$  be a set. The functor PairsOf  $G$  yielding a subset of  $G$  is defined as follows:

- (Def. 1) For every set  $e$  holds  $e \in \text{PairsOf } G$  iff  $e \in G$  and  $\overline{e} = 2$ .

The following propositions are true:

- (11) For every set  $X$  and for every set  $e$  such that  $e \in \text{PairsOf } X$  there exist sets  $x, y$  such that  $x \neq y$  and  $x, y \in \bigcup X$  and  $e = \{x, y\}$ .
- (12) For all sets  $X, x, y$  such that  $x \neq y$  and  $\{x, y\} \in X$  holds  $\{x, y\} \in \text{PairsOf } X$ .
- (13) For all sets  $X, x, y$  such that  $\{x, y\} \in \text{PairsOf } X$  holds  $x \neq y$  and  $x, y \in \bigcup X$ .
- (14) For all sets  $G, H$  such that  $G \subseteq H$  holds  $\text{PairsOf } G \subseteq \text{PairsOf } H$ .
- (15) For every finite set  $X$  holds  $\overline{\overline{\{\{x, \langle y, \bigcup X \rangle\}; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X: \{x, y\} \in \text{PairsOf } X\}}} = 2 \cdot \overline{\overline{\text{PairsOf } X}}$ .
- (16) For every finite set  $X$  holds  $\overline{\overline{\{\{x, y\}; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X: \{x, y\} \in \text{PairsOf } X\}}} = 2 \cdot \overline{\overline{\text{PairsOf } X}}$ .

Let  $X$  be a finite set. Note that PairsOf  $X$  is finite.

Let  $X$  be a set. We say that  $X$  is void if and only if:

- (Def. 2)  $X = \{\emptyset\}$ .

One can verify that there exists a set which is void.

Let us observe that every set which is void is also finite.

Let  $G$  be a void set. Observe that  $\bigcup G$  is empty.

Next we state two propositions:

- (17) For every set  $X$  such that  $X$  is void holds  $\text{PairsOf } X = \emptyset$ .
- (18) For every set  $X$  such that  $\bigcup X = \emptyset$  holds  $X = \emptyset$  or  $X = \{\emptyset\}$ .

Let  $X$  be a set. We say that  $X$  is pair free if and only if:

(Def. 3) PairsOf  $X$  is empty.

We now state the proposition

(19) For all sets  $X, x$  such that  $\overline{\bigcup X} = 1$  holds  $X$  is pair free.

Let us observe that there exists a set which is finite-membered and non empty.

Let  $X$  be a finite-membered set and let  $Y$  be a set. Observe that  $X \cap Y$  is finite-membered and  $X \setminus Y$  is finite-membered.

## 2. SIMPLE GRAPHS AS SIMPLICIAL COMPLEXES

Let  $n$  be a natural number and let  $X$  be a set. We say that  $X$  is at most  $n$ -dimensional if and only if:

(Def. 4) For every set  $x$  such that  $x \in X$  holds  $\overline{x} \subseteq n + 1$ .

Let  $n$  be a natural number. Observe that every set which is at most  $n$ -dimensional is also finite-membered.

Let  $n$  be a natural number. Observe that there exists a set which is at most  $n$ -dimensional, subset-closed, and non empty.

Next we state two propositions:

(20) For every subset-closed non empty set  $G$  holds  $\emptyset \in G$ .

(21) Let  $n$  be a natural number,  $X$  be an at most  $n$ -dimensional set, and  $x$  be an element of  $X$ . If  $x \in X$ , then  $\overline{x} \leq n + 1$ .

Let  $n$  be a natural number and let  $X, Y$  be at most  $n$ -dimensional sets. Note that  $X \cup Y$  is at most  $n$ -dimensional.

Let  $n$  be a natural number, let  $X$  be an at most  $n$ -dimensional set, and let  $Y$  be a set. Note that  $X \cap Y$  is at most  $n$ -dimensional and  $X \setminus Y$  is at most  $n$ -dimensional.

Let  $n$  be a natural number and let  $X$  be an at most  $n$ -dimensional set. Observe that every at most  $n$ -dimensional set is at most  $n$ -dimensional.

Let  $s$  be a set. We say that  $s$  is simple graph-like if and only if:

(Def. 5)  $s$  is at most 1-dimensional, subset-closed, and non empty.

Let us note that every set which is simple graph-like is also at most 1-dimensional, subset-closed, and non empty and every set which is at most 1-dimensional, subset-closed, and non empty is also simple graph-like.

The following proposition is true

(22)  $\{\emptyset\}$  is simple graph-like.

One can verify that  $\{\emptyset\}$  is simple graph-like.

One can verify that there exists a set which is simple graph-like.

A simple graph is a simple graph-like set.

One can verify that there exists a simple graph which is void and there exists a simple graph which is finite.

Let  $G$  be a set. We introduce Vertices  $G$  as a synonym of  $\bigcup G$ . We introduce Edges  $G$  as a synonym of PairsOf  $G$ .

Let  $X$  be a set. We introduce  $X$  is edgesless as a synonym of  $X$  is pair free.

We now state three propositions:

- (23) For every simple graph  $G$  such that Vertices  $G$  is finite holds  $G$  is finite.
- (24) For every simple graph  $G$  and for every set  $x$  holds  $x \in \text{Vertices } G$  iff  $\{x\} \in G$ .
- (25) For every set  $x$  holds  $\{\emptyset, \{x\}\}$  is a simple graph.

Let  $X$  be a finite finite-membered set. The functor order  $X$  yielding a natural number is defined by:

(Def. 6) order  $X = \overline{\bigcup X}$ .

Let  $X$  be a finite set. The functor size  $X$  yielding a natural number is defined by:

(Def. 7) size  $X = \overline{\overline{\text{PairsOf } X}}$ .

Next we state the proposition

- (26) For every finite simple graph  $G$  holds order  $G \leq \overline{G}$ .

Let  $G$  be a simple graph. A vertex of  $G$  is an element of Vertices  $G$ . An edge of  $G$  is an element of Edges  $G$ .

The following propositions are true:

- (27) For every simple graph  $G$  holds  $G = \{\emptyset\} \cup S_{(\text{Vertices } G)} \cup \text{Edges } G$ .
- (28) For every simple graph  $G$  such that Vertices  $G = \emptyset$  holds  $G$  is void.
- (29) Let  $G$  be a simple graph and  $x$  be a set. If  $x \in G$  and  $x \neq \emptyset$ , then there exists a set  $y$  such that  $x = \{y\}$  and  $y \in \text{Vertices } G$  or  $x \in \text{Edges } G$ .
- (30) For every simple graph  $G$  and for every set  $x$  such that Vertices  $G = \{x\}$  holds  $G = \{\emptyset, \{x\}\}$ .
- (31) For every set  $X$  there exists a simple graph  $G$  such that  $G$  is edgesless and Vertices  $G = X$ .

Let  $G$  be a simple graph and let  $v$  be an element of Vertices  $G$ . The functor Adjacent( $v$ ) yielding a subset of Vertices  $G$  is defined by:

(Def. 8) For every element  $x$  of Vertices  $G$  holds  $x \in \text{Adjacent}(v)$  iff  $\{v, x\} \in \text{Edges } G$ .

Let  $X$  be a set. A simple graph is called a simple graph of  $X$  if:

(Def. 9) Vertices it =  $X$ .

Let  $X$  be a set. The functor CompleteSGraph  $X$  yields a simple graph of  $X$  and is defined by:

(Def. 10) CompleteSGraph  $X = \{V; V \text{ ranges over finite subsets of } X: \overline{V} \leq 2\}$ .

One can prove the following proposition

- (32) For every simple graph  $G$  such that for all sets  $x, y$  such that  $x, y \in \text{Vertices } G$  holds  $\{x, y\} \in G$  holds  $G = \text{CompleteSGraph } \text{Vertices } G$ .

Let  $X$  be a finite set. One can check that  $\text{CompleteSGraph } X$  is finite.

The following propositions are true:

- (33) For every set  $X$  and for every set  $x$  such that  $x \in X$  holds  $\{x\} \in \text{CompleteSGraph } X$ .
- (34) For every set  $X$  and for all sets  $x, y$  such that  $x, y \in X$  holds  $\{x, y\} \in \text{CompleteSGraph } X$ .
- (35)  $\text{CompleteSGraph } \emptyset = \{\emptyset\}$ .
- (36) For every set  $x$  holds  $\text{CompleteSGraph } \{x\} = \{\emptyset, \{x\}\}$ .
- (37) For all sets  $x, y$  holds  $\text{CompleteSGraph } \{x, y\} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ .
- (38) For all sets  $X, Y$  such that  $X \subseteq Y$  holds  $\text{CompleteSGraph } X \subseteq \text{CompleteSGraph } Y$ .
- (39) For every simple graph  $G$  and for every set  $x$  such that  $x \in \text{Vertices } G$  holds  $\text{CompleteSGraph } \{x\} \subseteq G$ .

Let  $G$  be a simple graph. One can check that there exists a subset of  $G$  which is simple graph-like.

Let  $G$  be a simple graph. A subgraph of  $G$  is a simple graph-like subset of  $G$ .

Let  $G$  be a simple graph. The functor  $\text{Complement } G$  yields a simple graph and is defined as follows:

- (Def. 11)  $\text{Complement } G = \text{CompleteSGraph } \text{Vertices } G \setminus \text{Edges } G$ .

Let us observe that the functor  $\text{Complement } G$  is involutive.

Next we state two propositions:

- (40) For every simple graph  $G$  holds  $\text{Vertices } G = \text{Vertices } \text{Complement } G$ .
- (41) Let  $G$  be a simple graph and  $x, y$  be sets. If  $x \neq y$  and  $x, y \in \text{Vertices } G$ , then  $\{x, y\} \in \text{Edges } G$  iff  $\{x, y\} \notin \text{Edges } \text{Complement } G$ .

### 3. INDUCED SUBGRAPHS

Let  $G$  be a simple graph and let  $L$  be a set. The subgraph induced by  $G$  yielding a subset of  $G$  is defined by:

- (Def. 12) The subgraph induced by  $G = G \cap 2^L$ .

Let  $G$  be a simple graph and let  $L$  be a set. Observe that the subgraph induced by  $G$  is simple graph-like.

Next we state two propositions:

- (42) For every simple graph  $G$  holds  $G =$  the subgraph induced by  $G$ .
- (43) For every simple graph  $G$  and for every set  $L$  holds the subgraph induced by  $G =$  the subgraph induced by  $G$ .

Let  $G$  be a finite simple graph and let  $L$  be a set. Observe that the subgraph induced by  $G$  is finite.

Let  $G$  be a simple graph and let  $L$  be a finite set. One can check that the subgraph induced by  $G$  is finite.

One can prove the following three propositions:

- (44) For all simple graphs  $G, H$  such that  $G \subseteq H$  holds  $G \subseteq$  the subgraph induced by  $H$ .
- (45) For every simple graph  $G$  and for every set  $L$  holds  $\text{Vertices (the subgraph induced by } G) = \text{Vertices } G \cap L$ .
- (46) For every simple graph  $G$  and for every set  $x$  such that  $x \in \text{Vertices } G$  holds the subgraph induced by  $G = \{\emptyset, \{x\}\}$ .

#### 4. CLIQUE, CLIQUE NUMBER, CLIQUE COVER

Let  $G$  be a simple graph. We say that  $G$  is a clique if and only if:

(Def. 13)  $G = \text{CompleteSGraph Vertices } G$ .

The following propositions are true:

- (47) Let  $G$  be a simple graph. Suppose that for all sets  $x, y$  such that  $x \neq y$  and  $x, y \in \text{Vertices } G$  holds  $\{x, y\} \in \text{Edges } G$ . Then  $G$  is a clique.
- (48)  $\{\emptyset\}$  is a clique.

Observe that there exists a simple graph which is a clique. Let  $G$  be a simple graph. Note that there exists a subgraph of  $G$  which is a clique.

Let  $G$  be a simple graph. A clique of  $G$  is a clique subgraph of  $G$ .

Next we state the proposition

- (49) For every set  $X$  holds  $\text{CompleteSGraph } X$  is a clique.

Let  $X$  be a set. One can check that  $\text{CompleteSGraph } X$  is a clique.

Next we state two propositions:

- (50) For every simple graph  $G$  and for every set  $x$  such that  $x \in \text{Vertices } G$  holds  $\{\emptyset, \{x\}\}$  is a clique of  $G$ .
- (51) Let  $G$  be a simple graph and  $x, y$  be sets. If  $x, y \in \text{Vertices } G$  and  $\{x, y\} \in G$ , then  $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$  is a clique of  $G$ .

Let  $G$  be a simple graph. Observe that there exists a clique of  $G$  which is finite.

We now state two propositions:

- (52) For every simple graph  $G$  and for every set  $x$  such that  $x \in \bigcup G$  there exists a finite clique  $C$  of  $G$  such that  $\text{Vertices } C = \{x\}$ .
- (53) For every a clique simple graph  $C$  and for all sets  $u, v$  such that  $u, v \in \text{Vertices } C$  holds  $\{u, v\} \in C$ .

Let  $G$  be a simple graph. We say that  $G$  has finite clique number if and only if:

(Def. 14) There exists a finite clique  $C$  of  $G$  such that for every finite clique  $D$  of  $G$  holds order  $D \leq$  order  $C$ .

Let us note that there exists a simple graph which has finite clique number.

Let us observe that every simple graph which is finite also has finite clique number.

Let  $G$  be a simple graph with finite clique number. The functor  $\omega(G)$  yielding a natural number is defined as follows:

(Def. 15) There exists a finite clique  $C$  of  $G$  such that order  $C = \omega(G)$  and for every finite clique  $T$  of  $G$  holds order  $T \leq \omega(G)$ .

We now state several propositions:

(54) For every simple graph  $G$  with finite clique number such that  $\omega(G) = 0$  holds Vertices  $G = \emptyset$ .

(55) For every void simple graph  $G$  holds  $\omega(G) = 0$ .

(56) Let  $G$  be a simple graph and  $x, y$  be sets. If  $\{x, y\} \in G$ , then the subgraph induced by  $G$  is a clique of  $G$ .

(57) For every simple graph  $G$  with finite clique number such that Edges  $G \neq \emptyset$  holds  $\omega(G) \geq 2$ .

(58) For all simple graphs  $G, H$  with finite clique number such that  $G \subseteq H$  holds  $\omega(G) \leq \omega(H)$ .

(59) For every finite set  $X$  holds  $\omega(\text{CompleteSGraph } X) = \overline{\overline{X}}$ .

Let  $G$  be a simple graph and let  $P$  be a partition of Vertices  $G$ . We say that  $P$  is clique-wise if and only if:

(Def. 16) For every set  $x$  such that  $x \in P$  holds the subgraph induced by  $G$  is a clique of  $G$ .

Let  $G$  be a simple graph. Observe that there exists a partition of Vertices  $G$  which is clique-wise.

Let  $G$  be a simple graph. A clique-partition of  $G$  is a clique-wise partition of Vertices  $G$ .

Let  $G$  be a void simple graph. Note that every partition of Vertices  $G$  which is empty is also clique-wise.

Let  $G$  be a simple graph. We say that  $G$  has finite clique cover if and only if:

(Def. 17) There exists a clique-partition of  $G$  which is finite.

One can verify that every simple graph which is finite also has finite clique cover.

Let  $G$  be a simple graph with finite clique cover. Note that there exists a clique-partition of  $G$  which is finite.

Let  $G$  be a simple graph with finite clique cover and let  $S$  be a subset of Vertices  $G$ . One can verify that the subgraph induced by  $G$  has finite clique cover.

Let  $G$  be a simple graph with finite clique cover. The functor  $\kappa(G)$  yielding a natural number is defined by:

(Def. 18) There exists a finite clique-partition  $C$  of  $G$  such that  $\overline{\overline{C}} = \kappa(G)$  and for every finite clique-partition  $C$  of  $G$  holds  $\kappa(G) \leq \overline{\overline{C}}$ .

## 5. STABLE SET, COLORING

Let  $G$  be a simple graph and let  $S$  be a subset of Vertices  $G$ . We say that  $S$  is stable if and only if:

(Def. 19) For all sets  $x, y$  such that  $x \neq y$  and  $x, y \in S$  holds  $\{x, y\} \notin G$ .

We now state two propositions:

(60) For every simple graph  $G$  holds  $\emptyset_{\text{Vertices } G}$  is stable.

(61) For every simple graph  $G$  and for every subset  $S$  of Vertices  $G$  and for every set  $v$  such that  $S = \{v\}$  holds  $S$  is stable.

Let  $G$  be a simple graph. Observe that every subset of Vertices  $G$  which is trivial is also stable.

Let  $G$  be a simple graph. Note that there exists a subset of Vertices  $G$  which is stable.

Let  $G$  be a simple graph. A stable set of  $G$  is a stable subset of Vertices  $G$ .

The following two propositions are true:

(62) For every simple graph  $G$  and for all sets  $x, y$  such that  $x, y \in \text{Vertices } G$  and  $\{x, y\} \notin G$  holds  $\{x, y\}$  is a stable set of  $G$ .

(63) For every simple graph  $G$  with finite clique number such that  $\omega(G) = 1$  holds Vertices  $G$  is a stable set of  $G$ .

Let  $G$  be a simple graph. Note that there exists a stable set of  $G$  which is finite.

One can prove the following proposition

(64) For every simple graph  $G$  and for every stable set  $A$  of  $G$  holds every subset of  $A$  is a stable set of  $G$ .

Let  $G$  be a simple graph and let  $P$  be a partition of Vertices  $G$ . We say that  $P$  is stable-wise if and only if:

(Def. 20) For every set  $x$  such that  $x \in P$  holds  $x$  is a stable set of  $G$ .

The following proposition is true

(65) For every simple graph  $G$  holds  $\text{SmallestPartition}(\text{Vertices } G)$  is stable-wise.



Let  $G$  be a simple graph. Note that there exists a partition of Vertices  $G$  which is stable-wise. A coloring of  $G$  is a stable-wise partition of Vertices  $G$ . We say that  $G$  is finitely colorable if and only if:

(Def. 21) There exists a coloring of  $G$  which is finite.

One can verify that there exists a simple graph which is finitely colorable.

Let us note that every simple graph which is finite is also finitely colorable.

Let  $G$  be a finitely colorable simple graph. Note that there exists a coloring of  $G$  which is finite.

We now state two propositions:

(66) Let  $G$  be a simple graph,  $S$  be a clique of  $G$ , and  $L$  be a set. If  $L \subseteq$  Vertices  $S$ , then the subgraph induced by  $G$  is a clique of  $G$ .

(67) Let  $G$  be a simple graph,  $C$  be a coloring of  $G$ , and  $S$  be a subset of Vertices  $G$ . Then  $C|_S$  is a coloring of the subgraph induced by  $G$ .

Let  $G$  be a finitely colorable simple graph and let  $S$  be a set. One can check that the subgraph induced by  $G$  is finitely colorable. The functor  $\chi(G)$  yielding a natural number is defined as follows:

(Def. 22) There exists a finite coloring  $C$  of  $G$  such that  $\overline{C} = \chi(G)$  and for every finite coloring  $C$  of  $G$  holds  $\chi(G) \leq \overline{C}$ .

One can prove the following three propositions:

(68) For all finitely colorable simple graphs  $G, H$  such that  $G \subseteq H$  holds  $\chi(G) \leq \chi(H)$ .

(69) For every finite set  $X$  holds  $\chi(\text{CompleteSGraph } X) = \overline{X}$ .

(70) Let  $G$  be a finitely colorable simple graph,  $C$  be a finite coloring of  $G$ , and  $c$  be a set. Suppose  $c \in C$  and  $\overline{C} = \chi(G)$ . Then there exists an element  $v$  of Vertices  $G$  such that  $v \in c$  and for every element  $d$  of  $C$  such that  $d \neq c$  there exists an element  $w$  of Vertices  $G$  such that  $w \in \text{Adjacent}(v)$  and  $w \in d$ .

Let  $G$  be a simple graph. We say that  $G$  has finite stability number if and only if:

(Def. 23) There exists a finite stable set  $A$  of  $G$  such that for every finite stable set  $B$  of  $G$  holds  $\overline{B} \leq \overline{A}$ .

One can check that every simple graph which is finite also has finite stability number.

Let  $G$  be a simple graph with finite stability number. Observe that every stable set of  $G$  is finite.

Let us note that there exists a simple graph which is non void and has finite stability number.

Let  $G$  be a simple graph with finite stability number. The functor  $\alpha(G)$  yielding a natural number is defined as follows:

(Def. 24) There exists a finite stable set  $A$  of  $G$  such that  $\overline{A} = \alpha(G)$  and for every finite stable set  $T$  of  $G$  holds  $\overline{T} \leq \alpha(G)$ .

Let  $G$  be a non void simple graph with finite stability number. One can check that  $\alpha(G)$  is positive.

Next we state the proposition

(71) For every simple graph  $G$  with finite stability number such that  $\alpha(G) = 1$  holds  $G$  is a clique.

Let us observe that every simple graph which has finite clique number and finite stability number is also finite.

We now state four propositions:

(72) For every simple graph  $G$  and for every clique  $C$  of  $G$  holds Vertices  $C$  is a stable set of Complement  $G$ .

(73) For every simple graph  $G$  and for every clique  $C$  of Complement  $G$  holds Vertices  $C$  is a stable set of  $G$ .

(74) For every simple graph  $G$  and for every stable set  $C$  of  $G$  holds the subgraph induced by Complement  $G$  is a clique of Complement  $G$ .

(75) For every simple graph  $G$  and for every stable set  $C$  of Complement  $G$  holds the subgraph induced by  $G$  is a clique of  $G$ .

Let  $G$  be a simple graph with finite clique number. One can check that Complement  $G$  has finite stability number.

Let  $G$  be a simple graph with finite stability number. Note that Complement  $G$  has finite clique number.

We now state several propositions:

(76) For every simple graph  $G$  with finite clique number holds  $\omega(G) = \alpha(\text{Complement } G)$ .

(77) For every simple graph  $G$  with finite stability number holds  $\alpha(G) = \omega(\text{Complement } G)$ .

(78) For every simple graph  $G$  holds every clique-partition of Complement  $G$  is a coloring of  $G$ .

(79) For every simple graph  $G$  holds every clique-partition of  $G$  is a coloring of Complement  $G$ .

(80) For every simple graph  $G$  holds every coloring of  $G$  is a clique-partition of Complement  $G$ .

(81) For every simple graph  $G$  holds every coloring of Complement  $G$  is a clique-partition of  $G$ .

Let  $G$  be a finitely colorable simple graph. One can check that Complement  $G$  has finite clique cover.

Let  $G$  be a simple graph with finite clique cover.

One can check that Complement  $G$  is finitely colorable.

One can prove the following propositions:

- (82) For every finitely colorable simple graph  $G$  holds  $\chi(G) = \kappa(\text{Complement } G)$ .
- (83) For every simple graph  $G$  with finite clique cover holds  $\kappa(G) = \chi(\text{Complement } G)$ .

## 6. MYCIELSKIAN OF A GRAPH

Let  $G$  be a simple graph. The functor Mycielskian  $G$  yielding a simple graph is defined by the condition (Def. 25).

(Def. 25) Mycielskian  $G = \{\emptyset\} \cup \{\{x\} : x \text{ ranges over elements of } \bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}\} \cup \text{Edges } G \cup \{\{x, \langle y, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G, y \text{ ranges over elements of } \bigcup G : \{x, y\} \in \text{Edges } G\} \cup \{\{\bigcup G, \langle x, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G : x \in \text{Vertices } G\}$ .

We now state several propositions:

- (84) For every simple graph  $G$  holds  $G \subseteq \text{Mycielskian } G$ .
- (85) Let  $G$  be a simple graph and  $v$  be a set. Then  $v \in \text{Vertices Mycielskian } G$  if and only if one of the following conditions is satisfied:
- (i)  $v \in \bigcup G$ , or
  - (ii) there exists a set  $x$  such that  $x \in \bigcup G$  and  $v = \langle x, \bigcup G \rangle$ , or
  - (iii)  $v = \bigcup G$ .
- (86) For every simple graph  $G$  holds  $\text{Vertices Mycielskian } G = \bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}$ .
- (87) For every simple graph  $G$  holds  $\bigcup G \in \bigcup \text{Mycielskian } G$ .
- (88) For every void simple graph  $G$  holds  $\text{Mycielskian } G = \{\emptyset, \{\bigcup G\}\}$ .

Let  $G$  be a finite simple graph. Note that Mycielskian  $G$  is finite.

The following propositions are true:

- (89) For every finite simple graph  $G$  holds  $\text{order Mycielskian } G = 2 \cdot \text{order } G + 1$ .
- (90) Let  $G$  be a simple graph and  $e$  be a set. Then  $e \in \text{Edges Mycielskian } G$  if and only if one of the following conditions is satisfied:
- (i)  $e \in \text{Edges } G$ , or
  - (ii) there exist elements  $x, y$  of  $\bigcup G$  such that  $e = \{x, \langle y, \bigcup G \rangle\}$  and  $\{x, y\} \in \text{Edges } G$ , or
  - (iii) there exists an element  $y$  of  $\bigcup G$  such that  $e = \{\bigcup G, \langle y, \bigcup G \rangle\}$  and  $y \in \bigcup G$ .
- (91) Let  $G$  be a simple graph. Then  $\text{Edges Mycielskian } G = \text{Edges } G \cup \{\{x, \langle y, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G, y \text{ ranges over elements of } \bigcup G : \{x, y\} \in \text{Edges } G\} \cup \{\{\bigcup G, \langle y, \bigcup G \rangle\} : y \text{ ranges over elements of } \bigcup G : y \in \bigcup G\}$ .

- (92) For every finite simple graph  $G$  holds  $\text{size Mycielskian } G = 3 \cdot \text{size } G + \text{order } G$ .
- (93) Let  $G$  be a simple graph and  $s, t$  be sets. Suppose  $\{s, t\} \in \text{Edges Mycielskian } G$ . Then
- (i)  $\{s, t\} \in \text{Edges } G$ , or
  - (ii)  $s \in \bigcup G$  or  $s = \bigcup G$  but there exists a set  $y$  such that  $y \in \bigcup G$  and  $t = \langle y, \bigcup G \rangle$ , or
  - (iii)  $t \in \bigcup G$  or  $t = \bigcup G$  but there exists a set  $y$  such that  $y \in \bigcup G$  and  $s = \langle y, \bigcup G \rangle$ .
- (94) For every simple graph  $G$  and for every set  $u$  such that  $\{\bigcup G, u\} \in \text{Edges Mycielskian } G$  there exists a set  $x$  such that  $x \in \bigcup G$  and  $u = \langle x, \bigcup G \rangle$ .
- (95) For every simple graph  $G$  and for every set  $u$  such that  $u \in \text{Vertices } G$  holds  $\{\langle u, \bigcup G \rangle\} \in \text{Mycielskian } G$ .
- (96) For every simple graph  $G$  and for every set  $u$  such that  $u \in \text{Vertices } G$  holds  $\{\langle u, \bigcup G \rangle, \bigcup G\} \in \text{Mycielskian } G$ .
- (97) For every simple graph  $G$  and for all sets  $x, y$  holds  $\{\langle x, \bigcup G \rangle, \langle y, \bigcup G \rangle\} \notin \text{Edges Mycielskian } G$ .
- (98) For every simple graph  $G$  and for all sets  $x, y$  such that  $x \neq y$  holds  $\{\langle x, \bigcup G \rangle, \langle y, \bigcup G \rangle\} \notin \text{Mycielskian } G$ .
- (99) For every simple graph  $G$  and for all sets  $x, y$  such that  $\{\langle x, \bigcup G \rangle, y\} \in \text{Edges Mycielskian } G$  holds  $x \neq y$  but  $x \in \bigcup G$  but  $y \in \bigcup G$  or  $y = \bigcup G$ .
- (100) For every simple graph  $G$  and for all sets  $x, y$  such that  $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$  holds  $x \neq y$ .
- (101) For every simple graph  $G$  and for all sets  $x, y$  such that  $y \in \bigcup G$  and  $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$  holds  $\{x, y\} \in G$ .
- (102) For every simple graph  $G$  and for all sets  $x, y$  such that  $\{x, y\} \in \text{Edges } G$  holds  $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$ .
- (103) For every simple graph  $G$  and for all sets  $x, y$  such that  $x, y \in \text{Vertices } G$  and  $\{x, y\} \in \text{Mycielskian } G$  holds  $\{x, y\} \in G$ .
- (104) For every simple graph  $G$  holds  $G =$  the subgraph induced by Mycielskian  $G$ .
- (105) Let  $G$  be a simple graph and  $C$  be a finite clique of Mycielskian  $G$ . If  $3 \leq \text{order } C$ , then for every vertex  $v$  of  $C$  holds  $v \neq \bigcup G$ .
- (106) For every simple graph  $G$  with finite clique number such that  $\omega(G) = 0$  and for every finite clique  $D$  of Mycielskian  $G$  holds  $\text{order } D \leq 1$ .
- (107) For every simple graph  $G$  and for every set  $x$  such that  $\text{Vertices } G = \{x\}$  holds  $\text{Mycielskian } G = \{\emptyset, \{x\}, \{\langle x, \bigcup G \rangle\}, \{\bigcup G\}, \{\langle x, \bigcup G \rangle, \bigcup G\}\}$ .
- (108) For every simple graph  $G$  with finite clique number such that  $\omega(G) = 1$

and for every finite clique  $D$  of Mycielskian  $G$  holds order  $D \leq 2$ .

- (109) For every simple graph  $G$  with finite clique number such that  $2 \leq \omega(G)$  and for every finite clique  $D$  of Mycielskian  $G$  holds order  $D \leq \omega(G)$ .

Let  $G$  be a simple graph with finite clique number. Note that Mycielskian  $G$  has finite clique number.

We now state two propositions:

- (110) For every simple graph  $G$  with finite clique number such that  $2 \leq \omega(G)$  holds  $\omega(\text{Mycielskian } G) = \omega(G)$ .

- (111) For every finitely colorable simple graph  $G$  there exists a coloring  $E$  of Mycielskian  $G$  such that  $\overline{E} = 1 + \chi(G)$ .

Let  $G$  be a finitely colorable simple graph. Observe that Mycielskian  $G$  is finitely colorable.

We now state the proposition

- (112) For every finitely colorable simple graph  $G$  holds  $\chi(\text{Mycielskian } G) = 1 + \chi(G)$ .

Let  $G$  be a simple graph. The Mycielskian sequence of  $G$  yields a many sorted set indexed by  $\mathbb{N}$  and is defined by the condition (Def. 26).

(Def. 26) There exists a function  $m_1$  such that

- (i) the Mycielskian sequence of  $G = m_1$ ,
- (ii)  $m_1(0) = G$ , and
- (iii) for every natural number  $k$  and for every simple graph  $G$  such that  $G = m_1(k)$  holds  $m_1(k+1) = \text{Mycielskian } G$ .

We now state two propositions:

- (113) For every simple graph  $G$  holds (the Mycielskian sequence of  $G$ )(0) =  $G$ .

- (114) Let  $G$  be a simple graph and  $n$  be a natural number. Then (the Mycielskian sequence of  $G$ )( $n$ ) is a simple graph.

Let  $G$  be a simple graph and let  $n$  be a natural number. Observe that (the Mycielskian sequence of  $G$ )( $n$ ) is simple graph-like.

The following proposition is true

- (115) Let  $G, H$  be simple graphs and  $n$  be a natural number. Then (the Mycielskian sequence of  $G$ )( $n+1$ ) = Mycielskian (the Mycielskian sequence of  $G$ )( $n$ ).

Let  $G$  be a simple graph with finite clique number and let  $n$  be a natural number. One can check that (the Mycielskian sequence of  $G$ )( $n$ ) has finite clique number.

Let  $G$  be a finitely colorable simple graph and let  $n$  be a natural number. One can check that (the Mycielskian sequence of  $G$ )( $n$ ) is finitely colorable.

Let  $G$  be a finite simple graph and let  $n$  be a natural number. Observe that (the Mycielskian sequence of  $G$ )( $n$ ) is finite.

One can prove the following propositions:

- (116) Let  $G$  be a finite simple graph and  $n$  be a natural number. Then order (the Mycielskian sequence of  $G$ )( $n$ ) =  $(2^n \cdot \text{order } G + 2^n) - 1$ .
- (117) Let  $G$  be a finite simple graph and  $n$  be a natural number. Then size (the Mycielskian sequence of  $G$ )( $n$ ) =  $3^n \cdot \text{size } G + (3^n - 2^n) \cdot \text{order } G + ((n + 1) \text{ block } 3)$ .
- (118) Let  $n$  be a natural number. Then  $\omega$ ((the Mycielskian sequence of CompleteSGraph 2)( $n$ )) = 2 and  $\chi$ ((the Mycielskian sequence of CompleteSGraph 2)( $n$ )) =  $n + 2$ .
- (119) For every natural number  $n$  there exists a finite simple graph  $G$  such that  $\omega(G) = 2$  and  $\chi(G) > n$ .
- (120) For every natural number  $n$  there exists a finite simple graph  $G$  such that  $\alpha(G) = 2$  and  $\kappa(G) > n$ .

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