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Sorting by Exchanging

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Summary. We show that exchanging of pairs in an array which are in incorrect order leads to sorted array. It justifies correctness of Bubble Sort, Insertion Sort, and Quicksort.

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The notation and terminology used here have been introduced in the following papers: [20], [6], [11], [1], [8], [16], [12], [13], [10], [9], [17], [18], [3], [4], [2], [7], [14], [21], [22], [19], [5], and [15].

1. Preliminaries

We adopt the following convention: α, β, γ, δ denote ordinal numbers, k denotes a natural number, and x, y, z, t, X, Y, Z denote sets.

The following propositions are true:

(1) $x \in (\alpha + \beta) \setminus \alpha$ iff there exists γ such that $x = \alpha + \gamma$ and $\gamma \in \beta$.

(2) Suppose $\alpha \in \beta$ and $\gamma \in \delta$. Then $\gamma \neq \alpha$ and $\gamma \neq \beta$ and $\delta \neq \alpha$ and $\delta \neq \beta$ or $\gamma \in \alpha$ and $\delta = \alpha$ or $\gamma \in \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\delta \in \beta$ or $\gamma = \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\delta = \beta$ or $\gamma = \beta$ and $\beta \in \delta$.

(3) If $x \notin y$, then $(y \cup \{x\}) \setminus y = \{x\}$.

(4) $\operatorname{succ} x \setminus x = \{x\}$.

(5) Let f be a function, r be a binary relation, and given x. Then $x \in f^\circ r$ if and only if there exist y, z such that $\langle y, z \rangle \in r$ and $\langle y, z \rangle \in \operatorname{dom} f$ and $f(y, z) = x$.

(6) If $\alpha \setminus \beta \neq \emptyset$, then $\inf(\alpha \setminus \beta) = \beta$ and $\sup(\alpha \setminus \beta) = \alpha$ and $\bigcup(\alpha \setminus \beta) = \bigcup \alpha$.

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(7) If $\alpha \setminus \beta$ is non empty and finite, then there exists a natural number n such that $\alpha = \beta + n$.

2. Arrays

Let f be a set. We say that f is segmental if and only if:

(Def. 1) There exist α, β such that $\pi_1(f) = \alpha \setminus \beta$.

In the sequel f, g denote functions.

The following two propositions are true:

(8) If $\operatorname{dom} f = \operatorname{dom} g$ and f is segmental, then g is segmental.

(9) If f is segmental, then for all α, β, γ such that $\alpha \subseteq \beta \subseteq \gamma$ and α, $\gamma \in \operatorname{dom} f$ holds $\beta \in \operatorname{dom} f$.

Let us observe that every function which is transfinite sequence-like is also segmental and every function which is finite sequence-like is also segmental.

Let us consider α and let s be a set. We say that s is α-based if and only if:

(Def. 2) If $\beta \in \pi_1(s)$, then $\alpha \in \pi_1(s)$ and $\alpha \subseteq \beta$.

We say that s is α-limited if and only if:

(Def. 3) $\alpha = \sup \pi_1(s)$.

Next we state two propositions:

(10) f is α-based and segmental iff there exists β such that $\operatorname{dom} f = \beta \setminus \alpha$ and $\alpha \subseteq \beta$.

(11) f is β-limited, non empty, and segmental iff there exists α such that $\operatorname{dom} f = \beta \setminus \alpha$ and $\alpha \in \beta$.

Let us observe that every function which is transfinite sequence-like is also 0-based and every function which is finite sequence-like is also 1-based.

The following three propositions are true:

(12) f is $\inf \operatorname{dom} f$-based.

(13) f is $\sup \operatorname{dom} f$-limited.

(14) If f is β-limited and $\alpha \in \operatorname{dom} f$, then $\alpha \in \beta$.

Let us consider f. The functor $\operatorname{base} f$ yielding an ordinal number is defined as follows:

(Def. 4)(i) f is $\operatorname{base} f$-based if there exists α such that $\alpha \in \operatorname{dom} f$,

(ii) $\operatorname{base} f = 0$, otherwise.

The functor $\operatorname{limit} f$ yields an ordinal number and is defined as follows:

(Def. 5)(i) f is $\operatorname{limit} f$-limited if there exists α such that $\alpha \in \operatorname{dom} f$,

(ii) $\operatorname{limit} f = 0$, otherwise.

Let us consider f. The functor $\operatorname{length} f$ yielding an ordinal number is defined as follows:

(Def. 6) length f = limit f − base f.

We now state four propositions:

(15) base $\emptyset = 0$ and limit $\emptyset = 0$ and length $\emptyset = 0$.

(16) limit f = sup dom f.

(17) f is limit f-limited.

(18) Every empty set is α-based.

Let us consider α, X, Y. Note that there exists a transfinite sequence which is Y-defined, X-valued, α-based, segmental, finite, and empty.

An array is a segmental function.

Let A be an array. Observe that dom A is ordinal-membered.

We now state the proposition

(19) For every array f holds f is 0-limited iff f is empty.

Let us mention that every array which is 0-based is also transfinite sequence-like.

Let us consider X. An array of X is a X-valued array.

Let X be a 1-sorted structure. An array of X is an array of the carrier of X.

Let us consider α, X. An array of α, X is a α-defined array of X.

In the sequel A, B, C denote arrays.

Next we state several propositions:

(20) base f = inf dom f.

(21) f is base f-based.

(22) dom A = limit $A \setminus$ base A.

(23) If dom $A = \alpha \setminus \beta$ and A is non empty, then base $A = \beta$ and limit $A = \alpha$.

(24) For every transfinite sequence f holds base $f = 0$ and limit f = dom f and length f = dom f.

Let us consider α, β, X. Note that there exists an array of α, X which is β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite.

Let us consider α, x. Note that $\{\langle \alpha, x \rangle\}$ is segmental.

Let us consider α and let x be a natural number. Observe that $\{\langle \alpha, x \rangle\}$ is natural-valued.

Let us consider α and let x be a real number. One can verify that $\{\langle \alpha, x \rangle\}$ is real-valued.

Let us consider α, let X be a non empty set, and let x be an element of X. One can check that $\{\langle \alpha, x \rangle\}$ is X-valued.

Let us consider α, x. One can check that $\{\langle \alpha, x \rangle\}$ is α-based and succ α-limited.

Let us consider β. Note that there exists an array which is non empty, β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite. Let X be a non empty set. Note that there exists an array which is non empty, β-based, finite, and X-valued.

Let s be a transfinite sequence. We introduce $s\,\mathrm{last}$ as a synonym of $\mathrm{last}\, s$.

Let A be an array. The functor $\mathrm{last}\, A$ is defined by:

(Def. 7) $\mathrm{last}\, A = A(\bigcup \mathrm{dom}\, A)$.

3. Descending Sequences

Let f be a function. We say that f is descending if and only if:

(Def. 8) For all α, β such that α, $\beta \in \mathrm{dom}\, f$ and $\alpha \in \beta$ holds $f(\beta) \subset f(\alpha)$.

We now state four propositions:

(25) For every finite array f such that for every α such that α, $\mathrm{succ}\, \alpha \in \mathrm{dom}\, f$ holds $f(\mathrm{succ}\, \alpha) \subset f(\alpha)$ holds f is descending.

(26) For every array f such that $\mathrm{length}\, f = \omega$ and for every α such that α, $\mathrm{succ}\, \alpha \in \mathrm{dom}\, f$ holds $f(\mathrm{succ}\, \alpha) \subset f(\alpha)$ holds f is descending.

(27) For every transfinite sequence f such that f is descending and $f(0)$ is finite holds f is finite.

(28) Let f be a transfinite sequence. Suppose f is descending and $f(0)$ is finite and for every α such that $f(\alpha) \neq \emptyset$ holds $\mathrm{succ}\, \alpha \in \mathrm{dom}\, f$. Then $\mathrm{last}\, f = \emptyset$.

The scheme A deals with a transfinite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{F}(\mathcal{A}(0))$ is finite, and
- For every α such that $\mathrm{succ}\, \alpha \in \mathrm{dom}\, \mathcal{A}$ and $\mathcal{F}(\mathcal{A}(\alpha))$ is finite holds $\mathcal{F}(\mathcal{A}(\mathrm{succ}\, \alpha)) \subset \mathcal{F}(\mathcal{A}(\alpha))$.

4. Swap

Let us consider X, let f be a X-defined function, and let α, β be sets. Note that $\mathrm{Swap}(f, \alpha, \beta)$ is X-defined.

Let X be a set, let f be a X-valued function, and let x, y be sets. Note that $\mathrm{Swap}(f, x, y)$ is X-valued.

The following propositions are true:

(29) If x, $y \in \mathrm{dom}\, f$, then $(\mathrm{Swap}(f, x, y))(x) = f(y)$.

(30) For every array f of X such that x, $y \in \mathrm{dom}\, f$ holds $(\mathrm{Swap}(f, x, y))_x = f_y$.

(31) If x, $y \in \mathrm{dom}\, f$, then $(\mathrm{Swap}(f, x, y))(y) = f(x)$.

(32) For every array f of X such that x, $y \in \mathrm{dom}\, f$ holds $(\mathrm{Swap}(f, x, y))_y = f_x$.

(33) If $z \neq x$ and $z \neq y$, then $(\mathrm{Swap}(f, x, y))(z) = f(z)$.

(34) For every array f of X such that $z \in \mathrm{dom}\, f$ and $z \neq x$ and $z \neq y$ holds $(\mathrm{Swap}(f, x, y))_z = f_z$.

(35) If x, $y \in \mathrm{dom}\, f$, then $\mathrm{Swap}(f, x, y) = \mathrm{Swap}(f, y, x)$.

Let X be a non empty set. Observe that there exists a X-valued non empty function which is onto.

Let X be a non empty set, let f be an onto X-valued non empty function, and let x, y be elements of $\mathrm{dom}\, f$. Note that $\mathrm{Swap}(f, x, y)$ is onto.

Let us consider A and let us consider x, y. Note that $\mathrm{Swap}(A, x, y)$ is segmental.

We now state the proposition

(36) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(\mathrm{Swap}(A, x, y), x, y) = A$.

Let A be a real-valued array and let us consider x, y. One can verify that $\mathrm{Swap}(A, x, y)$ is real-valued.

5. Permutations

Let A be an array. An array is called a permutation of A if:

(Def. 9) There exists a permutation f of $\mathrm{dom}\, A$ such that it $= A \cdot f$.

We now state several propositions:

(37) For every permutation B of A holds $\mathrm{dom}\, B = \mathrm{dom}\, A$ and $\mathrm{rng}\, B = \mathrm{rng}\, A$.

(38) A is a permutation of A.

(39) If A is a permutation of B, then B is a permutation of A.

(40) If A is a permutation of B and B is a permutation of C, then A is a permutation of C.

(41) $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one, X-valued, and X-defined.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is onto and total.

Let X, Y be non empty sets, let f be a function from X into Y, and let x, y be elements of X. Then $\mathrm{Swap}(f, x, y)$ is a function from X into Y.

Next we state three propositions:

(42) If x, $y \in X$ and $f = \mathrm{Swap}(\mathrm{id}_X, x, y)$ and $X = \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y) = A \cdot f$.

(43) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y)$ is a permutation of A and A is a permutation of $\mathrm{Swap}(A, x, y)$.

(44) Suppose x, $y \in \mathrm{dom}\, A$ and A is a permutation of B. Then $\mathrm{Swap}(A, x, y)$ is a permutation of B and A is a permutation of $\mathrm{Swap}(B, x, y)$.

6. Exchanging

Let O be a relational structure and let A be an array of O. We say that A is ascending if and only if:

(Def. 10) For all α, β such that $\alpha, \beta \in \operatorname{dom} A$ and $\alpha \in \beta$ holds $A_\alpha \leq A_\beta$.

The functor inversions A is defined by:

(Def. 11) inversions $A = \{\langle\alpha, \beta\rangle; \alpha$ ranges over elements of $\operatorname{dom} A, \beta$ ranges over elements of $\operatorname{dom} A : \alpha \in \beta \ \wedge \ A_\alpha \not\leq A_\beta\}$.

Let O be a relational structure. One can verify that every empty array of O is ascending. Let A be an empty array of O. One can verify that inversions A is empty.

In the sequel O is a connected non empty poset and R, Q are arrays of O.

We now state the proposition

(45) For every O and for all elements x, y of O holds $x > y$ iff $x \not\leq y$.

Let us consider O, R. Then inversions R can be characterized by the condition:

(Def. 12) inversions $R = \{\langle\alpha, \beta\rangle; \alpha$ ranges over elements of $\operatorname{dom} R, \beta$ ranges over elements of $\operatorname{dom} R : \alpha \in \beta \ \wedge \ R_\alpha > R_\beta\}$.

Next we state two propositions:

(46) $\langle x, y\rangle \in$ inversions R iff $x, y \in \operatorname{dom} R$ and $x \in y$ and $R_x > R_y$.

(47) inversions $R \subseteq \operatorname{dom} R \times \operatorname{dom} R$.

Let us consider O and let R be a finite array of O. Observe that inversions R is finite.

We now state three propositions:

(48) R is ascending iff inversions $R = \emptyset$.

(49) If $\langle x, y\rangle \in$ inversions R, then $\langle y, x\rangle \notin$ inversions R.

(50) If $\langle x, y\rangle$, $\langle y, z\rangle \in$ inversions R, then $\langle x, z\rangle \in$ inversions R.

Let us consider O, R. Note that inversions R is relation-like.

Let us consider O, R. Note that inversions R is asymmetric and transitive.

Let us consider O and let α, β be elements of O. Let us note that the predicate $\alpha < \beta$ is antisymmetric.

Next we state several propositions:

(51) If $\langle x, y\rangle \in$ inversions R, then $\langle x, y\rangle \notin$ inversions $\operatorname{Swap}(R, x, y)$.

(52) If $x, y \in \operatorname{dom} R$ and $z \neq x$ and $z \neq y$ and $t \neq x$ and $t \neq y$, then $\langle z, t\rangle \in$ inversions R iff $\langle z, t\rangle \in$ inversions $\operatorname{Swap}(R, x, y)$.

(53) If $\langle x, y\rangle \in$ inversions R, then $\langle z, y\rangle \in$ inversions R and $z \in x$ iff $\langle z, x\rangle \in$ inversions $\operatorname{Swap}(R, x, y)$.

(54) If $\langle x, y\rangle \in$ inversions R, then $\langle z, x\rangle \in$ inversions R iff $z \in x$ and $\langle z, y\rangle \in$ inversions $\operatorname{Swap}(R, x, y)$.

(55) If $\langle x, y\rangle \in$ inversions R and $z \in y$ and $\langle x, z\rangle \in$ inversions $\mathrm{Swap}(R, x, y)$, then $\langle x, z\rangle \in$ inversions R.

(56) If $\langle x, y\rangle \in$ inversions R and $x \in z$ and $\langle z, y\rangle \in$ inversions $\mathrm{Swap}(R, x, y)$, then $\langle z, y\rangle \in$ inversions R.

(57) If $\langle x, y\rangle \in$ inversions R and $y \in z$ and $\langle x, z\rangle \in$ inversions $\mathrm{Swap}(R, x, y)$, then $\langle y, z\rangle \in$ inversions R.

(58) If $\langle x, y\rangle \in$ inversions R, then $y \in z$ and $\langle x, z\rangle \in$ inversions R iff $\langle y, z\rangle \in$ inversions $\mathrm{Swap}(R, x, y)$.

Let us consider O, R, x, y. The functor $\subseteq^{R}_{x,y}$ yields a function and is defined by:

(Def. 13) $\subseteq^{R}_{x,y} = \mathrm{Swap}(\mathrm{id}_{\mathrm{dom}\,R}, x, y) \times \mathrm{Swap}(\mathrm{id}_{\mathrm{dom}\,R}, x, y) +\cdot\, \mathrm{id}_{\{x\}\times(\mathrm{succ}\,y\setminus x)\cup(\mathrm{succ}\,y\setminus x)\times\{y\}}$.

Next we state the proposition

(59) $\gamma \in \mathrm{succ}\,\beta \setminus \alpha$ iff $\alpha \subseteq \gamma \subseteq \beta$.

We adopt the following convention: T is a non empty array of O and p, q, r, s are elements of $\mathrm{dom}\,T$.

The following propositions are true:

(60) $\mathrm{succ}\,q \setminus p \subseteq \mathrm{dom}\,T$.

(61) $\mathrm{dom} \subseteq^{T}_{p,q} = \mathrm{dom}\,T \times \mathrm{dom}\,T$ and $\mathrm{rng} \subseteq^{T}_{p,q} \subseteq \mathrm{dom}\,T \times \mathrm{dom}\,T$.

(62) If $p \subseteq r \subseteq q$, then $(\subseteq^{T}_{p,q})(p, r) = \langle p, r\rangle$ and $(\subseteq^{T}_{p,q})(r, q) = \langle r, q\rangle$.

(63) If $r \neq p$ and $s \neq q$ and $f = \mathrm{Swap}(\mathrm{id}_{\mathrm{dom}\,T}, p, q)$, then $(\subseteq^{T}_{p,q})(r, s) = \langle f(r), f(s)\rangle$.

(64) If $r \in p$ and $f = \mathrm{Swap}(\mathrm{id}_{\mathrm{dom}\,T}, p, q)$, then $(\subseteq^{T}_{p,q})(r, q) = \langle f(r), f(q)\rangle$ and $(\subseteq^{T}_{p,q})(r, p) = \langle f(r), f(p)\rangle$.

(65) If $q \in r$ and $f = \mathrm{Swap}(\mathrm{id}_{\mathrm{dom}\,T}, p, q)$, then $(\subseteq^{T}_{p,q})(p, r) = \langle f(p), f(r)\rangle$ and $(\subseteq^{T}_{p,q})(q, r) = \langle f(q), f(r)\rangle$.

(66) If $p \in q$, then $(\subseteq^{T}_{p,q})(p, q) = \langle p, q\rangle$.

(67) If $p \in q$ and $r \neq p$ and $r \neq q$ and $s \neq p$ and $s \neq q$, then $(\subseteq^{T}_{p,q})(r, s) = \langle r, s\rangle$.

(68) If $r \in p$ and $p \in q$, then $(\subseteq^{T}_{p,q})(r, p) = \langle r, q\rangle$ and $(\subseteq^{T}_{p,q})(r, q) = \langle r, p\rangle$.

(69) If $p \in s$ and $s \in q$, then $(\subseteq^{T}_{p,q})(p, s) = \langle p, s\rangle$ and $(\subseteq^{T}_{p,q})(s, q) = \langle s, q\rangle$.

(70) If $p \in q$ and $q \in s$, then $(\subseteq^{T}_{p,q})(p, s) = \langle q, s\rangle$ and $(\subseteq^{T}_{p,q})(q, s) = \langle p, s\rangle$.

(71) If $p \in q$, then $\subseteq^{T}_{p,q} \restriction$(inversions $\mathrm{Swap}(T, p, q)$ **qua** set) is one-to-one.

Let us consider O, R, x, y, z. Note that $(\subseteq^{R}_{x,y})^{\circ} z$ is relation-like.

7. Correctness of Sorting by Exchanging

The following proposition is true

(72) If $\langle x, y\rangle \in \operatorname{inversions} R$, then $(\subseteq^{R}_{x,y})^{\circ} \operatorname{inversions} \operatorname{Swap}(R, x, y) \subset \operatorname{inversions} R$.

Let R be a finite function and let us consider x, y. One can check that $\operatorname{Swap}(R, x, y)$ is finite.

Next we state two propositions:

(73) For every array R of O such that $\langle x, y\rangle \in \operatorname{inversions} R$ and $\operatorname{inversions} R$ is finite holds $\overline{\overline{\operatorname{inversions} \operatorname{Swap}(R, x, y)}} \in \overline{\overline{\operatorname{inversions} R}}$.

(74) For every finite array R of O such that $\langle x, y\rangle \in \operatorname{inversions} R$ holds $\overline{\overline{\operatorname{inversions} \operatorname{Swap}(R, x, y)}} < \overline{\overline{\operatorname{inversions} R}}$.

Let us consider O, R. A non empty array is called a computation of R if it satisfies the conditions (Def. 14).

(Def. 14)(i) $\operatorname{It}(\operatorname{base} \operatorname{it}) = R$,

(ii) for every α such that $\alpha \in \operatorname{dom} \operatorname{it}$ holds $\operatorname{it}(\alpha)$ is an array of O, and

(iii) for every α such that α, $\operatorname{succ} \alpha \in \operatorname{dom} \operatorname{it}$ there exist R, x, y such that $\langle x, y\rangle \in \operatorname{inversions} R$ and $\operatorname{it}(\alpha) = R$ and $\operatorname{it}(\operatorname{succ} \alpha) = \operatorname{Swap}(R, x, y)$.

We now state the proposition

(75) $\{\langle \alpha, R\rangle\}$ is a computation of R.

Let us consider O, R, α. One can check that there exists a computation of R which is α-based and finite.

Let us consider O, R, let C be a computation of R, and let us consider x. One can check that $C(x)$ is segmental, function-like, and relation-like.

Let us consider O, R, let C be a computation of R, and let us consider x. Observe that $C(x)$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. Observe that $\operatorname{last} C$ is segmental, relation-like, and function-like.

Let us consider O, R and let C be a computation of R. Observe that $\operatorname{last} C$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. We say that C is complete if and only if:

(Def. 15) $\operatorname{last} C$ is ascending.

One can prove the following three propositions:

(76) For every 0-based computation C of R such that R is a finite array of O holds C is finite.

(77) Let C be a 0-based computation of R. Suppose R is a finite array of O and for every α such that $\operatorname{inversions} C(\alpha) \neq \emptyset$ holds $\operatorname{succ} \alpha \in \operatorname{dom} C$. Then C is complete.

(78) Let C be a finite computation of R. Then $\operatorname{last} C$ is a permutation of R and for every α such that $\alpha \in \operatorname{dom} C$ holds $C(\alpha)$ is a permutation of R.

8. Existence of Complete Computations

Next we state three propositions:

(79) For every 0-based finite array A of X such that $A \neq \emptyset$ holds $\operatorname{last} A \in X$.

(80) $\operatorname{last}\langle x \rangle = x$.

(81) For every 0-based finite array A holds $\operatorname{last}(A \frown \langle x \rangle) = x$.

Let X be a set. Observe that every element of X^{ω} is X-valued.

The scheme A deals with a unary functor $\mathcal{F}$ yielding a set, a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite 0-based non empty array f and there exists an element k of $\mathcal{A}$ such that

(i) $k = \operatorname{last} f$,

(ii) $\mathcal{F}(k) = \emptyset$,

(iii) $f(0) = \mathcal{B}$, and

(iv) for every α such that $\operatorname{succ} \alpha \in \operatorname{dom} f$ there exist elements x, y of $\mathcal{A}$ such that $x = f(\alpha)$ and $y = f(\operatorname{succ} \alpha)$ and $\mathcal{P}[x, y]$

provided the following requirements are met:

- $\mathcal{B} \in \mathcal{A}$,
- $\mathcal{F}(\mathcal{B})$ is finite, and
- For every element x of $\mathcal{A}$ such that $\mathcal{F}(x) \neq \emptyset$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{F}(y) \subset \mathcal{F}(x)$.

In the sequel A is an array and B is a permutation of A.

We now state the proposition

(82) $B \in (\operatorname{rng} A)^{\operatorname{dom} A}$.

Let A be a real-valued array. One can verify that every permutation of A is real-valued.

Let us consider α and let X be a non empty set. Observe that every element of X^{α} is transfinite sequence-like.

Let us consider X and let Y be a real-membered non empty set. One can check that every element of Y^X is real-valued.

Let us consider X and let A be an array of X. One can check that every permutation of A is X-valued.

Let X be a set, let Z be a set, and let Y be a subset of Z. Note that every element of Y^X is Z-valued.

One can prove the following propositions:

(83) Every X-defined Y-valued binary relation is a relation between X and Y.

(84) For every finite ordinal number α and for every x such that $x \in \alpha$ holds $x = 0$ or there exists β such that $x = \operatorname{succ} \beta$.

(85) For every 0-based finite non empty array A of O holds there exists a 0-based computation of A which is complete.

(86) For every 0-based finite non empty array A of O holds there exists a permutation of A which is ascending.

Let us consider O and let A be a 0-based finite array of O. Observe that there exists a permutation of A which is ascending.

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