

Differentiable Functions into Real Normed Spaces

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Summary. In this article, we formalize the differentiability of functions from the set of real numbers into a normed vector space [14].

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The notation and terminology used here have been introduced in the following papers: [12], [2], [3], [7], [9], [11], [1], [4], [10], [13], [6], [17], [18], [15], [8], [16], [19], and [5].

For simplicity, we adopt the following rules: F denotes a non trivial real normed space, G denotes a real normed space, X denotes a set, x, x_0, r, p denote real numbers, n, k denote elements of \mathbb{N} , Y denotes a subset of \mathbb{R} , Z denotes an open subset of \mathbb{R} , s_1 denotes a sequence of real numbers, s_2 denotes a sequence of G , f, f_1, f_2 denote partial functions from \mathbb{R} to the carrier of F , h denotes a convergent to 0 sequence of real numbers, and c denotes a constant sequence of real numbers.

We now state two propositions:

- (1) If for every n holds $\|s_2(n)\| \leq s_1(n)$ and s_1 is convergent and $\lim s_1 = 0$, then s_2 is convergent and $\lim s_2 = 0_G$.
- (2) $(s_1 \uparrow k)(s_2 \uparrow k) = (s_1 s_2) \uparrow k$.

Let us consider F and let I_1 be a partial function from \mathbb{R} to the carrier of F . We say that I_1 is rest-like if and only if:

(Def. 1) I_1 is total and for every h holds $h^{-1}(I_{1*}h)$ is convergent and $\lim(h^{-1}(I_{1*}h)) = 0_F$.

Let us consider F . One can check that there exists a partial function from \mathbb{R} to the carrier of F which is rest-like. Let us consider F . A rest of F is a rest-like partial function from \mathbb{R} to the carrier of F . Let us consider F and let I_1 be a function from \mathbb{R} into the carrier of F . We say that I_1 is linear if and only if:

(Def. 2) There exists a point r of F such that for every real number p holds $I_1(p) = p \cdot r$.

Let us consider F . Note that there exists a function from \mathbb{R} into the carrier of F which is linear. Let us consider F . A linear of F is a linear function from \mathbb{R} into the carrier of F .

We use the following convention: R, R_1, R_2 denote rests of F and L, L_1, L_2 denote linears of F .

The following propositions are true:

- (3) $L_1 + L_2$ is a linear of F and $L_1 - L_2$ is a linear of F .
- (4) rL is a linear of F .
- (5) Let h_1, h_2 be partial functions from \mathbb{R} to the carrier of F and s_2 be a sequence of real numbers. If $\text{rng } s_2 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2)_*s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_*s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.
- (6) Let h_1, h_2 be partial functions from \mathbb{R} to the carrier of F and s_2 be a sequence of real numbers. If h_1 is total and h_2 is total, then $(h_1 + h_2)_*s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_*s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.
- (7) $R_1 + R_2$ is a rest of F and $R_1 - R_2$ is a rest of F .
- (8) rR is a rest of F .

Let us consider F, f and let x_0 be a real number. We say that f is differentiable in x_0 if and only if:

(Def. 3) There exists a neighbourhood N of x_0 such that $N \subseteq \text{dom } f$ and there exist L, R such that for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider F, f and let x_0 be a real number. Let us assume that f is differentiable in x_0 . The functor $f'(x_0)$ yielding a point of F is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood N of x_0 such that $N \subseteq \text{dom } f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider F, f, X . We say that f is differentiable on X if and only if:

(Def. 5) $X \subseteq \text{dom } f$ and for every x such that $x \in X$ holds $f|X$ is differentiable in x .

The following propositions are true:

- (9) If f is differentiable on X , then X is a subset of \mathbb{R} .
- (10) f is differentiable on Z iff $Z \subseteq \text{dom } f$ and for every x such that $x \in Z$ holds f is differentiable in x .
- (11) If f is differentiable on Y , then Y is open.

Let us consider F, f, X . Let us assume that f is differentiable on X . The functor $f'_{\upharpoonright X}$ yields a partial function from \mathbb{R} to the carrier of F and is defined by:

(Def. 6) $\text{dom}(f'_{\upharpoonright X}) = X$ and for every x such that $x \in X$ holds $f'_{\upharpoonright X}(x) = f'(x)$.

Next we state a number of propositions:

- (12) Suppose $Z \subseteq \text{dom } f$ and there exists a point r of F such that $\text{rng } f = \{r\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\upharpoonright Z})_x = 0_F$.
- (13) Let x_0 be a real number and N be a neighbourhood of x_0 . Suppose f is differentiable in x_0 and $N \subseteq \text{dom } f$. Let given h, c . Suppose $\text{rng } c = \{x_0\}$ and $\text{rng}(h + c) \subseteq N$. Then $h^{-1}((f_*(h + c)) - (f_*c))$ is convergent and $f'(x_0) = \lim(h^{-1}((f_*(h + c)) - (f_*c)))$.
- (14) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0 , then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$.
- (15) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0 , then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$.
- (16) For every real number r such that f is differentiable in x_0 holds $r f$ is differentiable in x_0 and $(r f)'(x_0) = r \cdot f'(x_0)$.
- (17) Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z . Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\upharpoonright Z}(x) = f_1'(x) + f_2'(x)$.
- (18) Suppose $Z \subseteq \text{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z . Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{\upharpoonright Z}(x) = f_1'(x) - f_2'(x)$.
- (19) Suppose $Z \subseteq \text{dom}(r f)$ and f is differentiable on Z . Then $r f$ is differentiable on Z and for every x such that $x \in Z$ holds $(r f)'_{\upharpoonright Z}(x) = r \cdot f'(x)$.
- (20) If $Z \subseteq \text{dom } f$ and $f|_Z$ is constant, then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\upharpoonright Z}(x) = 0_F$.
- (21) Let r, p be points of F and given Z, f . Suppose $Z \subseteq \text{dom } f$ and for every x such that $x \in Z$ holds $f_x = x \cdot r + p$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\upharpoonright Z}(x) = r$.
- (22) For every real number x_0 such that f is differentiable in x_0 holds f is continuous in x_0 .
- (23) If f is differentiable on X , then $f|_X$ is continuous.
- (24) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z .

(25) There exists a rest R of F such that $R_0 = 0_F$ and R is continuous in 0 .

Let us consider F and let f be a partial function from \mathbb{R} to the carrier of F . We say that f is differentiable if and only if:

(Def. 7) f is differentiable on $\text{dom } f$.

Let us consider F . One can check that there exists a function from \mathbb{R} into the carrier of F which is differentiable. We now state the proposition

(26) Let f be a differentiable partial function from \mathbb{R} to the carrier of F . If $Z \subseteq \text{dom } f$, then f is differentiable on Z .

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