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Tarski Geometry Axioms. Part V – Half-planes and Planes

Roland Coghetto
cafr-MSA2P asbl
Rue de la Brasserie 5
7100 La Louvière, Belgium

Adam Grabowski
Faculty of Computer Science
University of Białystok
Poland

Summary. In the article, we continue the formalization of the work devoted to Tarski's geometry – the book "Metamathematische Methoden in der Geometrie" by W. Schwabhäuser, W. Szmielew, and A. Tarski. We use the Mizar system to formalize Chapter 9 of this book. We deal with half-planes and planes proving their properties as well as the theory of intersecting lines.

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Introduction

In the article, we continue [6], [7], and [8] – the formalization of the work devoted to Tarski's geometry – the book "Metamathematische Methoden in der Geometrie" (SST for short) by W. Schwabhäuser, W. Szmielew, and A. Tarski [18], [10], [11]. We use the Mizar system [1], [2] to formalize (parts of) Chapter 9 of the SST book developing also results of Gupta [12] included there.

The first Mizar article formalizing Tarski's axioms [17] was inspired by another formalizations of SST: within the classical two-valued logic with Isabelle/HOL by Makarios [13, 14, 15], Metamath or by means of Coq [16, 4]. Some of the results were obtained with the help of other automatic proof assistants, either partially [9], or completely [3]. Relatively recent achievement was the import of huge portions of code from GeoCoq into Isabelle [5].

Here we define the notion of half-planes and planes and prove some of their basic properties, a theory of intersecting lines (including orthogonality), notions of betweenness including lines and points, shifting this notion into planes and spaces of higher dimension.

1. Preliminaries

Now we state the proposition:

(1) Let us consider Tarski plane S satisfying the axiom of congruence identity and the axiom of betweenness identity, and points a, b, c of S. If $a, b \leqslant c, c$, then $a = b$.

2. Betweenness Relation Revisited

Let S be a non empty Tarski plane, a, b be points of S, and A be a subset of S. We say that A lies between a and b if and only if

(Def. 1) A is a line and $a \notin A$ and $b \notin A$ and there exists a point t of S such that $t \in A$ and t lies between a and b.

Now we state the proposition:

(2) Let us consider a non empty Tarski plane S satisfying the axiom of betweenness identity, a point a of S, and a subset A of S. Then A does not lie between a and a.

Let S be a non empty Tarski plane and a, b, p, q be points of S. We say that between(a, p, q, b) if and only if

(Def. 2) $p \neq q$ and $\mathrm{Line}(p, q)$ lies between a and b.

From now on S denotes a non empty Tarski plane satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity, and the axiom of Pasch, a, b denote points of S, and A denotes a subset of S. Now we state the proposition:

(3) 9.2 Satz:
If A lies between a and b, then A lies between b and a.

In the sequel S denotes a non empty Tarski plane satisfying seven Tarski's geometry axioms, a, b, c, m, r, s denote points of S, and A denotes a subset of S. Now we state the propositions:

(4) If b lies between a and c and A is a line and a, $c \in A$, then $b \in A$.

(5) If b lies between a and c and $a \neq b$ and A is a line and a, $b \in A$, then $c \in A$.

(6) Suppose A lies between a and c and $m \in A$ and Middle(a, m, c) and $r \in A$. If $a \widetilde{\overline{\overline{r}}} b$ and b lies between r and a, then A lies between b and c. The theorem is a consequence of (4).

(7) 9.3 Lemma:
If A lies between a and c and $m \in A$ and Middle(a, m, c) and $r \in A$, then for every b such that $a \widetilde{\overline{\overline{r}}} b$ holds A lies between b and c. The theorem is a consequence of (6), (4), and (5).

Let S be a non empty Tarski plane satisfying seven Tarski's geometry axioms, a, b be points of S, and A be a subset of S. We say that $A \perp_a b$ if and only if

(Def. 3) $\overline{A, a} \perp \overline{a, b}$.

3. Half-lines and Outer Pasch

Let S be a non empty Tarski plane and K be a subset of S. We say that K is a half-line if and only if

(Def. 4) there exist points p, a of S such that $p \neq a$ and $K = \text{HalfLine}(p, a)$.

Now we state the proposition:

(8) Let us consider points a, b, c, d, e of S. Suppose $b \neq c$ and $c \neq d$ and c lies between b and d and (b lies between a and c or a lies between b and c) and (d lies between c and e or e lies between c and d). Then c lies between a and e.

From now on S denotes a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, a, b, c, d, m, p, q, r, s, x denote points of S, and A, A', E denote subsets of S. Now we state the propositions:

(9) Suppose $r \neq s$ and $s, c \leqslant r, a$ and A lies between a and c and $r \in A$ and $A \perp_r a$ and $s \in A$ and $A \perp_s c$. Then

(i) if Middle(r, m, s), then for every point u of S, $u \widetilde{\overline{\overline{r}}} a$ iff $\text{S}_m(u) \widetilde{\overline{\overline{s}}} c$, and

(ii) for every points u, v of S such that $u \widetilde{\overline{\overline{r}}} a$ and $v \widetilde{\overline{\overline{s}}} c$ holds A lies between u and v.

The theorem is a consequence of (1) and (7).

(10) 9.4 Lemma:
Suppose A lies between a and c and $r \in A$ and $A \perp_r a$ and $s \in A$ and $A \perp_s c$. Then

(i) if Middle(r, m, s), then for every point u of S, $u \widetilde{\overline{\overline{r}}} a$ iff $\text{S}_m(u) \widetilde{\overline{\overline{s}}} c$, and

(ii) for every points u, v of S such that $u \underset{r}{\approx} a$ and $v \underset{s}{\approx} c$ holds A lies between u and v.

The theorem is a consequence of (9) and (8).

(11) Let us consider points a, b of S. If $a \neq b$, then $b \underset{a}{\approx} b$.

(12) Satz 9.5 (Gupta 1965):
If A lies between a and c and $r \in A$, then for every b such that $a \underset{r}{\approx} b$ holds A lies between b and c.
Proof: Consider p, q being points of S such that $p \neq q$ and $A = \text{Line}(p, q)$. Consider x being a point of S such that x is perpendicular foot of p, q, a. $b \notin A$ by [7, (87), (45)]. Consider y being a point of S such that y is perpendicular foot of p, q, b. Consider z being a point of S such that z is perpendicular foot of p, q, c. Consider m being a point of S such that $\text{Middle}(x, m, z)$. Set $d = \text{S}_m(a)$. $d \notin A$ by [7, (87)]. $z \neq d$ by [7, (45), (87)]. $d \underset{z}{\approx} c$. A lies between a and d and $m \in A$ and $\text{Middle}(a, m, d)$ and $r \in A$ and $a \underset{r}{\approx} b$. A lies between b and d. □

(13) Satz 9.6 (Satz von Pasch, Exterior form – Gupta 1965):
If c lies between a and p and q lies between b and c, then there exists x such that x lies between a and b and q lies between p and x. The theorem is a consequence of (12).

4. Points on the Same Side of the Line

Let S be a non empty Tarski plane, A be a subset of S, and a, b be points of S. We say that $a \underset{A}{\approx} b$ if and only if

(Def. 5) there exists a point c of S such that A lies between a and c and A lies between b and c.

Let a, b, p, q be points of S. We say that $a \underset{p,q}{\approx} b$ if and only if

(Def. 6) $p \neq q$ and $a \underset{\text{Line}(p,q)}{\approx} b$.

Now we state the propositions:

(14) 9.8 Satz:
If A lies between a and c, then A lies between b and c iff $a \underset{A}{\approx} b$. The theorem is a consequence of (12).

(15) 9.9 Satz:
If A lies between a and b, then $\neg a \underset{A}{\approx} b$. The theorem is a consequence of (14).

(16) 9.10 Lemma:
If A is a line and $a \notin A$, then there exists c such that A lies between a and c.

PROOF: Consider p, q such that $p \neq q$ and $A = \mathrm{Line}(p, q)$.
Set $c = \mathrm{S}_p(a)$. $p \neq c$ by [7, (104)]. □

(17) 9.11 SATZ: REFLEXIVITY:
If A is a line and $a \notin A$, then $a \mathrel{\underset{A}{\approx}} a$. The theorem is a consequence of (16).

(18) 9.12 SATZ: SYMMETRY:
If $a \mathrel{\underset{A}{\approx}} b$, then $b \mathrel{\underset{A}{\approx}} a$.

(19) 9.13 SATZ: TRANSITIVITY:
If $a \mathrel{\underset{A}{\approx}} b$ and $b \mathrel{\underset{A}{\approx}} c$, then $a \mathrel{\underset{A}{\approx}} c$. The theorem is a consequence of (14).

5. HALF-PLANES

Let S be a non empty Tarski plane, A be a subset of S, and a be a point of S. The functor HalfPlane(A, a) yielding a subset of S is defined by the term

(Def. 7) $\{x$, where x is a point of $S : x \mathrel{\underset{A}{\approx}} a\}$.

Let S be a non empty Tarski plane and p, q, a be points of S. Assume p, q and a are not collinear. The functor HalfPlane(p, q, a) yielding a set is defined by the term

(Def. 8) HalfPlane(Line$(p, q), a)$.

Now we state the propositions:

(20) If A is a line and $a \notin A$, then $a \in$ HalfPlane(A, a). The theorem is a consequence of (17).

(21) If A is a line and $a \notin A$ and $b \notin A$ and $b \in$ HalfPlane(A, a), then $a \in$ HalfPlane(A, b).

(22) If $b \in$ HalfPlane(A, a), then HalfPlane$(A, b) \subseteq$ HalfPlane(A, a). The theorem is a consequence of (19).

(23) If A is a line and $a \notin A$ and $b \notin A$ and $b \in$ HalfPlane(A, a), then HalfPlane$(A, b) =$ HalfPlane(A, a). The theorem is a consequence of (21) and (22).

Let S be a non empty Tarski plane, A be a subset of S, and a, b be points of S. We say that a and b are on the opposite sides of A if and only if

(Def. 9) A lies between a and b.

Now we state the propositions:

(24) If $a \mathrel{\underset{A}{\approx}} b$, then A is a line and $a \notin A$ and $b \notin A$.

(25) 9.17 SATZ:
If $a \mathrel{\underset{A}{\approx}} b$ and c lies between a and b, then $c \mathrel{\underset{A}{\approx}} a$.
PROOF: Consider d being a point of S such that A lies between a and d and A lies between b and d. Consider x being a point of S such that $x \in A$

and x lies between a and d. Consider y being a point of S such that $y \in A$ and y lies between b and d. Consider t being a point of S such that t lies between c and d and t lies between x and y. $c \notin A$. A lies between c and d by (24), [7, (87), (14)]. □

6. Half-planes and Collinearity

Now we state the propositions:

(26) 9.18 Satz:
If A is a line and $p \in A$ and a, b and p are collinear, then A lies between a and b iff p lies between a and b and $a \notin A$ and $b \notin A$.

(27) If A is a line and $p \in A$ and $a \mathrel{\underset{p}{\cong}} b$ and $a \notin A$, then A lies between b and $\mathrm{S}_p(a)$.
Proof: Set $c = \mathrm{S}_p(a)$. p lies between a and c. $c \neq p$. $b \notin A$ by [7, (87), (73)]. $c \notin A$ by [7, (87)]. □

(28) If A is a line and $p \in A$ and $a \notin A$, then A lies between a and $\mathrm{S}_p(a)$.
Proof: Set $c = \mathrm{S}_p(a)$. p lies between a and c. $c \neq p$. $c \notin A$ by [7, (87)]. □

(29) 9.19 Satz:
If A is a line and $p \in A$ and a, b and p are collinear, then $a \mathrel{\underset{A}{\cong}} b$ iff $a \mathrel{\underset{p}{\cong}} b$ and $a \notin A$. The theorem is a consequence of (15), (28), and (27).

7. Planes

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, A be a subset of S, and r be a point of S. Assume A is a line and $r \notin A$. The functor $\mathrm{Plane}(A, r)$ yielding a subset of S is defined by

(Def. 10) there exists a point r' of S such that A lies between r and r' and $it = (\mathrm{HalfPlane}(A, r) \cup A) \cup \mathrm{HalfPlane}(A, r')$.

Now we state the propositions:

(30) If A is a line and $r \notin A$, then $\mathrm{HalfPlane}(A, r) \subseteq \mathrm{Plane}(A, r)$.

(31) If A is a line and $r \notin A$, then $A \subseteq \mathrm{Plane}(A, r)$ and $r \in \mathrm{Plane}(A, r)$. The theorem is a consequence of (20) and (30).

(32) Suppose A is a line and $r \notin A$. Then $\mathrm{Plane}(A, r) = \{x$, where x is a point of $S : x \mathrel{\underset{A}{\cong}} r$ or $x \in A$ or A lies between r and $x\}$.
Proof: Consider r' being a point of S such that A lies between r and r' and $\mathrm{Plane}(A, r) = (\mathrm{HalfPlane}(A, r) \cup A) \cup \mathrm{HalfPlane}(A, r')$. Set $P =$

$\{x$, where x is a point of $S : x \underset{A}{\simeq} r$ or $x \in A$ or A lies between r and $x\}$. Plane$(A, r) \subseteq P$ by [7, (14)], (14). $P \subseteq$ Plane(A, r) by [7, (14)]. □

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and p, q, r be points of S. Assume p, q and r are not collinear. The functor Plane(p, q, r) yielding a subset of S is defined by the term

(Def. 11) Plane(Line$(p, q), r)$.

Let E be a subset of S. We say that E is a plane if and only if

(Def. 12) there exist points p, q, r of S such that p, q and r are not collinear and $E =$ Plane(p, q, r).

Now we state the propositions:

(33) If A lies between a and b, then $b \in$ Plane(A, a). The theorem is a consequence of (32).

(34) 9.21 Satz:
If A is a line and $r \notin A$ and $s \in$ Plane(A, r) and $s \notin A$, then Plane$(A, r) =$ Plane(A, s). The theorem is a consequence of (14) and (23).

(35) If A, A' intersect at p and A, A' intersect at q, then $p = q$.

(36) If A is a line and a, $p \in A$, then $\mathrm{S}_p(a) \in A$.

(37) 9.22 Lemma:
If A, A' intersect at p and $r \in A'$ and $r \neq p$, then $A' \subseteq$ Plane(A, r). The theorem is a consequence of (32), (31), and (36).

(38) If A is a line and A' is a line and $A \neq A'$, then there exists a point r of S such that $r \notin A$ and $r \in A'$.

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and A, A' be subsets of S. Assume A is a line and A' is a line and $A \neq A'$ and $A \cap A'$ is not empty. The functor Plane(A, A') yielding a subset of S is defined by

(Def. 13) there exists a point r of S such that $r \notin A$ and $r \in A'$ and *it* $=$ Plane(A, r).

Now we state the propositions:

(39) Let us consider a non empty Tarski plane S, subsets A, B of S, and a point x of S. If A, B intersect at x, then B, A intersect at x.

(40) If A, A' intersect at p, then $A \subseteq$ Plane(A', A) and $A' \subseteq$ Plane(A, A'). The theorem is a consequence of (37).

(41) Suppose A, A' intersect at p. Then there exists a point r of S such that

(i) $r \notin A$, and

(ii) $r \in A'$, and

(iii) $\text{Plane}(A, A') = \text{Plane}(A, r)$, and

(iv) $A' = \text{Line}(r, p)$, and

(v) there exists a point r' of S such that p lies between r and r' and $p \neq r'$ and r, p and r' are collinear and $r' \notin A$ and $\text{Plane}(A, r) = \text{Plane}(A, r')$.

PROOF: Consider r being a point of S such that $r \notin A$ and $r \in A'$ and $\text{Plane}(A, A') = \text{Plane}(A, r)$. Consider r' being a point of S such that p lies between r and r' and $p \neq r'$. $r' \notin A$ by [7, (89)]. $r' \in A'$ and $A' \subseteq \text{Plane}(A, r)$. $\text{Plane}(A, r) = \text{Plane}(A, r')$. □

(42) If A, A' intersect at p, then $\text{Plane}(A, A') \subseteq \text{Plane}(A', A)$. The theorem is a consequence of (41), (32), (31), (40), (14), (34), (29), and (37).

Now we state the propositions:

(43) 9.24 SATZ:
If A, A' intersect at p, then $A \subseteq \text{Plane}(A, A')$ and $A' \subseteq \text{Plane}(A, A')$ and $\text{Plane}(A, A') = \text{Plane}(A', A)$. The theorem is a consequence of (39), (40), and (42).

(44) Suppose a, $b \in E$ and $a \neq b$ and p, q and r are not collinear and $E = \text{Plane}(p, q, r)$ and $c \in \text{Line}(p, q)$ and $c \notin \text{Line}(a, b)$ and $b \notin \text{Line}(p, q)$. Then

(i) $\text{Line}(a, b) \subseteq E$, and

(ii) there exists c such that a, b and c are not collinear and $E = \text{Plane}(a, b, c)$.

The theorem is a consequence of (43), (34), and (31).

(45) Suppose a, $b \in E$ and $a \neq b$ and p, q and r are not collinear and $E = \text{Plane}(p, q, r)$ and $b \notin \text{Line}(p, q)$ and $\text{Line}(p, q) \neq \text{Line}(a, b)$. Then

(i) $\text{Line}(a, b) \subseteq E$, and

(ii) there exists c such that a, b and c are not collinear and $E = \text{Plane}(a, b, c)$.

PROOF: Set $A = \text{Line}(p, q)$. Set $A' = \text{Line}(a, b)$. There exists a point c of S such that $c \notin A'$ and $c \in A$ by [7, (46), (83), (87)]. □

(46) SATZ 9.25:
If E is a plane and a, $b \in E$ and $a \neq b$, then $\text{Line}(a, b) \subseteq E$ and there exists c such that a, b and c are not collinear and $E = \text{Plane}(a, b, c)$. The theorem is a consequence of (31) and (45).

(47) SATZ 9.26:
If a, b and c are not collinear and E is a plane and a, b, $c \in E$, then $E = \text{Plane}(a, b, c)$. The theorem is a consequence of (46) and (34).

(48) If A is a line and $a \notin A$, then $a \in \mathrm{Plane}(A, a)$. The theorem is a consequence of (32) and (17).

(49) 9.27.(1) SATZ:
If a, b and c are not collinear, then there exists a subset E of S such that $\mathrm{Plane}(a, b, c) = E$ and E is a plane and a, b, $c \in E$. The theorem is a consequence of (31) and (48).

(50) 9.27.(2) SATZ:
If A is a line and $c \notin A$, then there exists a subset E of S such that E is a plane and $A \subseteq E$ and $c \in E$ and $\mathrm{Plane}(A, c) = E$. The theorem is a consequence of (31) and (48).

(51) 9.27.(3) SATZ:
If A, A' intersect at p, then there exists a subset E of S such that E is a plane and $A \subseteq E$ and $A' \subseteq E$ and $\mathrm{Plane}(A, A') = E$. The theorem is a consequence of (50) and (43).

(52) 9.28 FOLGERUNG:
Suppose a, b and c are not collinear. Let us consider subsets E_1, E_2 of S. Suppose E_1 is a plane and a, b, $c \in E_1$ and E_2 is a plane and a, b, $c \in E_2$. Then $E_1 = E_2$. The theorem is a consequence of (47).

(53) 9.29 FOLGERUNG:
Suppose a, b and c are not collinear. Then

(i) $\mathrm{Plane}(a, b, c) = \mathrm{Plane}(b, c, a)$, and

(ii) $\mathrm{Plane}(a, b, c) = \mathrm{Plane}(c, a, b)$, and

(iii) $\mathrm{Plane}(a, b, c) = \mathrm{Plane}(b, a, c)$, and

(iv) $\mathrm{Plane}(a, b, c) = \mathrm{Plane}(a, c, b)$, and

(v) $\mathrm{Plane}(a, b, c) = \mathrm{Plane}(c, b, a)$.

The theorem is a consequence of (49) and (52).

(54) 9.30 FOLGERUNG:
Suppose A is a line. Let us consider subsets E_1, E_2 of S. Suppose E_1 is a plane and E_2 is a plane and $A \subseteq E_1$ and $A \subseteq E_2$ and $E_1 \neq E_2$. Let us consider a point x of S. Then $x \in E_1$ and $x \in E_2$ if and only if $x \in A$. The theorem is a consequence of (52).

(55) If $s \underset{p,q}{\simeq} r$, then $s \neq p$ and $s \neq q$ and $r \neq p$ and $r \neq q$ and $p \neq q$.

(56) $\mathrm{Line}(b, c)$ does not lie between a and a.

(57) If A lies between a and b, then $a \neq b$.

(58) Let us consider Tarski plane S satisfying the axiom of congruence identity, the axiom of segment construction, the axiom of betweenness identity,

the axiom of Pasch, and Lower Dimension Axiom. Then there exist points p, q of S such that $p \neq q$.

(59) 9.31 Satz:
If $s \underset{p,q}{\simeq} r$ and $s \underset{p,r}{\simeq} q$, then $\mathrm{Line}(p, s)$ lies between q and r. The theorem is a consequence of (14), (29), (19), and (12).

8. Coplanarity Relation

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and A be a subset of S. We say that A is a set of coplanar points if and only if

(Def. 14) there exists a subset E of S such that E is a plane and $A \subseteq E$.

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and a, b, c, d be points of S. We say that a, b, c, d are coplanar if and only if

(Def. 15) there exists a subset E of S such that E is a plane and $a, b, c, d \in E$.

Now we state the propositions:

(60) Suppose a, b, c, d are coplanar. Then

(i) a, b, d, c are coplanar, and

(ii) a, c, b, d are coplanar, and

(iii) a, c, d, b are coplanar, and

(iv) a, d, c, b are coplanar, and

(v) a, d, b, c are coplanar, and

(vi) b, a, c, d are coplanar, and

(vii) b, a, d, c are coplanar, and

(viii) b, c, a, d are coplanar, and

(ix) b, c, d, a are coplanar, and

(x) b, d, a, c are coplanar, and

(xi) b, d, c, a are coplanar, and

(xii) c, a, b, d are coplanar, and

(xiii) c, a, d, b are coplanar, and

(xiv) c, b, a, d are coplanar, and

(xv) c, b, d, a are coplanar, and

(xvi) d, a, b, c are coplanar, and

(xvii) d, a, c, b are coplanar, and

(xviii) d, b, a, c are coplanar, and

(xix) d, b, c, a are coplanar.

(61) a, a, a, a are coplanar. The theorem is a consequence of (49).

(62) a, a, a, b are coplanar. The theorem is a consequence of (61) and (49).

(63) a, a, b, c are coplanar. The theorem is a consequence of (49), (46), and (62).

(64) If a, b and x are collinear and c, d and x are collinear and $a \neq x$ and $c \neq x$, then a, b, c, d are coplanar. The theorem is a consequence of (49), (31), and (53).

(65) If b, a and x are collinear and c, d and x are collinear and $b \neq x$ and $c \neq x$, then a, b, c, d are coplanar. The theorem is a consequence of (64).

(66) If a, b and x are collinear and c, d and x are collinear and $b \neq x$ and $c \neq x$, then a, b, c, d are coplanar. The theorem is a consequence of (65).

(67) Suppose a, b and x are collinear and c, d and x are collinear and ($b \neq x$ and $c \neq x$ or $b \neq x$ and $d \neq x$ or $a \neq x$ and $c \neq x$ or $a \neq x$ and $d \neq x$). Then a, b, c, d are coplanar. The theorem is a consequence of (66), (64), and (65).

(68) 9.33 Satz:
a, b, c, d are coplanar if and only if there exists x such that a, b and x are collinear and c, d and x are collinear or a, c and x are collinear and b, d and x are collinear or a, d and x are collinear and b, c and x are collinear. The theorem is a consequence of (63), (47), (53), (59), (32), and (67).

(69) Suppose a, b and c are not collinear. Then

(i) $\mathrm{Plane}(a, b, c)$ is a plane, and

(ii) a, b, $c \in \mathrm{Plane}(a, b, c)$, and

(iii) for every points u, v of S such that u, $v \in \mathrm{Plane}(a, b, c)$ and $u \neq v$ holds $\mathrm{Line}(u, v) \subseteq \mathrm{Plane}(a, b, c)$.

The theorem is a consequence of (49) and (46).

(70) 9.34 Satz:
Suppose a, b and c are not collinear. Let us consider a subset E of S. Suppose a, b, $c \in E$ and for every points u, v of S such that u, $v \in E$ and $u \neq v$ holds $\mathrm{Line}(u, v) \subseteq E$. Then $\mathrm{Plane}(a, b, c) \subseteq E$.
Proof: $\mathrm{Plane}(a, b, c)$ is a plane and a, b, $c \in \mathrm{Plane}(a, b, c)$ and for every points u, v of S such that u, $v \in \mathrm{Plane}(a, b, c)$ and $u \neq v$ holds $\mathrm{Line}(u, v) \subseteq \mathrm{Plane}(a, b, c)$. $a \neq c$ by [7, (46), (14)]. $b \neq c$ by [7, (46)]. $\mathrm{Plane}(a, b, c) \subseteq E$ by (68), [7, (14)]. □

9. Towards Higher Dimensions

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, a, b be points of S, and A be a subset of S. We say that between$^2(a, A, b)$ if and only if

(Def. 16) A is a plane and $a \notin A$ and $b \notin A$ and there exists a point t of S such that $t \in A$ and t lies between a and b.

Now we state the propositions:

(71) 9.38 Satz (n = 2):
If between$^2(a, A, b)$, then between$^2(b, A, a)$.

(72) If p lies between a and c and $a \underset{p}{\simeq} b$, then p lies between b and c.

(73) 9.39 Satz (n = 2):
If between$^2(a, A, c)$ and $r \in A$, then for every b such that $a \underset{r}{\simeq} b$ holds between$^2(b, A, c)$. The theorem is a consequence of (69) and (12).

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, a, b be points of S, and A be a subset of S. We say that $a \overset{2}{\underset{A}{\simeq}} b$ if and only if

(Def. 17) there exists a point c of S such that between$^2(a, A, c)$ and between$^2(b, A, c)$.

Now we state the propositions:

(74) 9.41 Satz (n = 2):
If between$^2(a, A, c)$, then between$^2(b, A, c)$ iff $a \overset{2}{\underset{A}{\simeq}} b$. The theorem is a consequence of (69) and (73).

(75) 9.9 Satz (Version n = 2):
If between$^2(a, A, b)$, then $\neg(a \overset{2}{\underset{A}{\simeq}} b)$. The theorem is a consequence of (74).

(76) 9.10 Lemma (Version n = 2):
If A is a plane and $a \notin A$, then there exists c such that between$^2(a, A, c)$.
Proof: Consider p, q, r such that p, q and r are not collinear and $A = \mathrm{Plane}(p, q, r)$. $r \notin \mathrm{Line}(p, q)$. $\mathrm{Line}(p, q) \subseteq A$. p, q, $r \in A$. Set $c = \mathrm{S}_p(a)$. $p \neq c$ by [7, (104)]. $c \notin A$. □

(77) 9.11 Satz (Version n = 2):
If A is a plane and $a \notin A$, then $a \overset{2}{\underset{A}{\simeq}} a$. The theorem is a consequence of (76).

(78) 9.12 Satz (Version n = 2):
If $a \overset{2}{\underset{A}{\simeq}} b$, then $b \overset{2}{\underset{A}{\simeq}} a$.

(79) 9.13 SATZ (VERSION N = 2):
If $a \overset{2}{\underset{A}{\approx}} b$ and $b \overset{2}{\underset{A}{\approx}} c$, then $a \overset{2}{\underset{A}{\approx}} c$. The theorem is a consequence of (74).

10. HALF-SPACES

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, A be a subset of S, and a be a point of S. Assume A is a plane and $a \notin A$. The functor $\mathrm{HalfSpace}^3(A, a)$ yielding a subset of S is defined by the term

(Def. 18) $\{x$, where x is a point of $S : x \overset{2}{\underset{A}{\approx}} a\}$.

Let p, q, a be points of S. Assume p, q and a are not collinear. The functor $\mathrm{HalfSpace}^3(p, q, a)$ yielding a set is defined by the term

(Def. 19) $\mathrm{HalfSpace}^3(\mathrm{Line}(p, q), a)$.

Now we state the propositions:

(80) If A is a plane and $a \notin A$, then $a \in \mathrm{HalfSpace}^3(A, a)$. The theorem is a consequence of (77).

(81) If A is a plane and $a \notin A$ and $b \notin A$ and $b \in \mathrm{HalfSpace}^3(A, a)$, then $a \in \mathrm{HalfSpace}^3(A, b)$.

(82) If A is a plane and $a \notin A$ and $b \notin A$ and $b \in \mathrm{HalfSpace}^3(A, a)$, then $\mathrm{HalfSpace}^3(A, b) \subseteq \mathrm{HalfSpace}^3(A, a)$. The theorem is a consequence of (79).

(83) If A is a plane and $a \notin A$ and $b \notin A$ and $b \in \mathrm{HalfSpace}^3(A, a)$, then $\mathrm{HalfSpace}^3(A, b) = \mathrm{HalfSpace}^3(A, a)$. The theorem is a consequence of (81) and (82).

11. TOWARDS SPACES IN HIGHER DIMENSIONS

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms, A be a subset of S, and r be a point of S. Assume A is a plane and $r \notin A$. The functor $\mathrm{Space}^3(A, r)$ yielding a subset of S is defined by

(Def. 20) there exists a point r' of S such that $\mathrm{between}^2(r, A, r')$ and $it = (\mathrm{HalfSpace}^3(A, r) \cup A) \cup \mathrm{HalfSpace}^3(A, r')$.

Now we state the propositions:

(84) If A is a plane and $r \notin A$, then $\mathrm{HalfSpace}^3(A, r) \subseteq \mathrm{Space}^3(A, r)$.

(85) If A is a plane and $r \notin A$, then $A \subseteq \mathrm{Space}^3(A, r)$ and $r \in \mathrm{Space}^3(A, r)$. The theorem is a consequence of (80) and (84).

(86) Suppose A is a plane and $r \notin A$. Then $\mathrm{Space}^3(A, r) = \{x$, where x is a point of $S : x \overset{2}{\underset{A}{\approx}} r$ or $x \in A$ or between$^2 (r, A, x)\}$.

PROOF: Consider r' being a point of S such that between$^2(r, A, r')$ and $\mathrm{Space}^3(A, r) = (\mathrm{HalfSpace}^3(A, r) \cup A) \cup \mathrm{HalfSpace}^3(A, r')$. Set $P = \{x$, where x is a point of $S : x \overset{2}{\underset{A}{\approx}} r$ or $x \in A$ or between$^2 (r, A, x)\}$. $\mathrm{Space}^3(A, r) \subseteq P$ by [7, (14)], (74). $P \subseteq \mathrm{Space}^3(A, r)$ by [7, (14)]. □

Let S be a non empty Tarski plane satisfying Lower Dimension Axiom and seven Tarski's geometry axioms and p_0, p_1, p_2, r be points of S. Assume p_0, p_1, p_2, r are not coplanar. The functor $\mathrm{Space}^3(p_0, p_1, p_2, r)$ yielding a subset of S is defined by the term

(Def. 21) $\mathrm{Space}^3(\mathrm{Plane}(p_0, p_1, p_2), r)$.

Let E be a subset of S. We say that E is a space^3 if and only if

(Def. 22) there exists a point r of S and there exists a subset A of S such that A is a plane and $r \notin A$ and $E = \mathrm{Space}^3(A, r)$.

Now we state the propositions:

(87) If A is a plane and a, b and c are not collinear and $a, b, c \in A$ and $d \notin A$, then a, b, c, d are not coplanar.

(88) Suppose E is a space^3. Then there exists a and there exists b and there exists c and there exists d such that a, b, c, d are not coplanar and $E = \mathrm{Space}^3(a, b, c, d)$. The theorem is a consequence of (69) and (87).

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