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# Integral of Continuous Functions of Two Variables[1]

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Yasunari Shidama
Karuizawa Hotch 244-1
Nagano, Japan

**Summary.** We extend the formalization of the integral theory of one-variable functions for Riemann and Lebesgue integrals, showing that the Lebesgue integral of a continuous function of two variables coincides with the Riemann iterated integral of a projective function.

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## Introduction

So far, the authors have proved in Mizar [2], [15] many theorems on the integral theory of one-variable functions for Riemann and Lebesgue integrals [9], [5], [11] (for interesting survey of formalizations of real analysis in another proof-assistants like ACL2 [13], Isabelle/HOL [12], Coq [3], see [4]). As a result, we have shown that if a function bounded on a closed interval (i.e., a continuous function) is Riemann integrable, then it is Lebesgue integrable, and both integrals coincide [10]. Furthermore, for the Lebesgue integral, there exist integral theorems on the product measure spaces [9]. From these results, this article

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shows that the Lebesgue integral of a continuous function of two variables coincides with the Riemann iterated integral of a projective function [1]. In the first three sections of this article, we summarize the basic properties of the projection of functions of two variables. In the last section, we prove integrability and iterated integrals of continuous functions of two variables.

Note that the continuity of functions of many variables is not directly addressed in this article, since there are quite a few formal notions of continuity which can be applied in this case (although they are essentially the same; for the discussion on the pros and cons of duplications in the Mizar Mathematical Library, see [14]). The formalization follows [19] and [16].

## 1. Preliminaries

Now we state the propositions:

(1) Let us consider a non empty set $X$, a $\sigma$-field $S$ of subsets of $X$, a $\sigma$-measure $M$ on $S$, and a partial function $f$ from $X$ to $\overline{\mathbb{R}}$. If $\operatorname{dom} f = \emptyset$, then $\int f \, \mathrm{d}M = 0$.

(2) Let us consider a non empty set $X$, a $\sigma$-field $S$ of subsets of $X$, a $\sigma$-measure $M$ on $S$, and a partial function $f$ from $X$ to $\mathbb{R}$. If $\operatorname{dom} f = \emptyset$, then $\int f \, \mathrm{d}M = 0$. The theorem is a consequence of (1).

(3) Let us consider a non empty set $X$, a $\sigma$-field $S$ of subsets of $X$, and a $\sigma$-measure $M$ on $S$. If $M$ is $\sigma$-finite, then $\mathrm{COM}(M)$ is $\sigma$-finite.

Proof: Consider $E$ being a set sequence of $S$ such that for every natural number $n$, $M(E(n)) < +\infty$ and $\bigcup E = X$. For every natural number $n$, $E(n) \in \mathrm{COM}(S, M)$. Reconsider $E_1 = E$ as a set sequence of $\mathrm{COM}(S, M)$. For every natural number $n$, $(\mathrm{COM}(M))(E_1(n)) < +\infty$. □

(4) B-Meas is $\sigma$-finite.

Proof: Define $\mathcal{S}(\text{natural number}) = [-\$_1, \$_1](\in 2^{\mathbb{R}})$. Consider $E$ being a function from $\mathbb{N}$ into $2^{\mathbb{R}}$ such that for every element $i$ of $\mathbb{N}$, $E(i) = \mathcal{S}(i)$. For every natural number $n$, $E(n) = [-n, n]$. For every natural number $n$, $E(n) \in$ the Borel sets by [7, (5)]. For every natural number $n$, $(\text{B-Meas})(E(n)) < +\infty$ by [8, (71)]. □

(5) L-Meas is $\sigma$-finite.

(6) ProdMeas(L-Meas, L-Meas) is $\sigma$-finite.

(7) Let us consider a closed interval subset $I$ of $\mathbb{R}$, and a subset $E$ of the real normed space of $\mathbb{R}$. If $I = E$, then $E$ is compact.

Proof: For every sequence $s_1$ of the real normed space of $\mathbb{R}$ such that $\operatorname{rng} s_1 \subseteq E$ there exists a sequence $s_2$ of the real normed space of $\mathbb{R}$ such that $s_2$ is subsequence of $s_1$ and convergent and $\lim s_2 \in E$. □

Let $S_1$, $S_2$ be real normed spaces, $D_1$ be a subset of $S_1$, and $D_2$ be a subset of $S_2$. Let us note that the functor $D_1 \times D_2$ yields a subset of $S_1 \times S_2$. Now we state the propositions:

(8) Let us consider real normed spaces $P$, $Q$, a subset $E$ of $P$, and a subset $F$ of $Q$. Suppose $E$ is compact and $F$ is compact. Then $E \times F$ is subset of $P \times Q$ and compact.
PROOF: Set $S = P \times Q$. Set $X = E \times F$. For every sequence $s_1$ of $S$ such that $\operatorname{rng} s_1 \subseteq X$ there exists a sequence $s_2$ of $S$ such that $s_2$ is subsequence of $s_1$ and convergent and $\lim s_2 \in X$. □

(9) Let us consider closed interval subsets $I$, $J$ of $\mathbb{R}$, and a subset $E$ of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$). If $E = I \times J$, then $E$ is compact. The theorem is a consequence of (7) and (8).

(10) Let us consider a set $E$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $f = g$ and $E \subseteq \operatorname{dom} f$. Then $f$ is uniformly continuous on $E$ if and only if for every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every real numbers $x_1$, $x_2$, $y_1$, $y_2$ such that $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle \in E$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ holds $|g(\langle x_2, y_2 \rangle) - g(\langle x_1, y_1 \rangle)| < e$.
PROOF: For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every points $z_1$, $z_2$ of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) such that $z_1$, $z_2 \in E$ and $\|z_1 - z_2\| < r$ holds $\|f_{/z_1} - f_{/z_2}\| < e$. □

(11) Let us consider intervals $I$, $J$. Then

   (i) $I \times J$ is a subset of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and

   (ii) $I \times J \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$.

(12) Let us consider a point $z$ of the real normed space of $\mathbb{R}$, and real numbers $x$, $r$. If $x = z$, then $\text{Ball}(z, r) = ]x - r, x + r[$.
PROOF: For every object $p$, $p \in \text{Ball}(z, r)$ iff $p \in ]x - r, x + r[$. □

(13) Let us consider a point $z$ of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and a real number $r$. Suppose $0 < r$. Then there exists a real number $s$ and there exist real numbers $x$, $y$ such that $0 < s < r$ and $z = \langle x, y \rangle$ and $]x - s, x + s[ \times ]y - s, y + s[ \subseteq \text{Ball}(z, r)$. The theorem is a consequence of (12).

Let us consider a subset $A$ of (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$). Now we state the propositions:

(14) Suppose for every real numbers $a$, $b$ such that $\langle a, b \rangle \in A$ there exists

a real-membered set $R$ such that $R$ is non empty and upper bounded and $R = \{r$,where $r$ is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[ \subseteq A\}$. Then there exists a function $F$ from $A$ into $\mathbb{R}$ such that for every real numbers $a$, $b$ such that $\langle a, b\rangle \in A$ there exists a real-membered set $R$ such that $R$ is non empty and upper bounded and $R = \{r$, where $r$ is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[ \subseteq A\}$ and $F(\langle a, b\rangle) = \frac{\sup R}{2}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist real numbers $a$, $b$ and there exists a real-membered set $R$ such that $\$_1 = \langle a, b\rangle$ and $R$ is non empty and upper bounded and $R = \{r$, where $r$ is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[ \subseteq A\}$ and $\$_2 = \frac{\sup R}{2}$. For every object $x$ such that $x \in A$ there exists an object $y$ such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$.

Consider $F$ being a function from $A$ into $\mathbb{R}$ such that for every object $x$ such that $x \in A$ holds $\mathcal{P}[x, F(x)]$. For every real numbers $a$, $b$ such that $\langle a, b\rangle \in A$ there exists a real-membered set $R$ such that $R$ is non empty and upper bounded and $R = \{r$, where $r$ is a real number : $0 < r$ and $]a-r, a+r[\times]b-r, b+r[ \subseteq A\}$ and $F(\langle a, b\rangle) = \frac{\sup R}{2}$. □

(15) If $A$ is open, then $A \in \sigma(\text{MeasRect}(\text{L-Field}, \text{L-Field}))$. The theorem is a consequence of (13) and (14).

(16) Let us consider a subset $H$ of the real normed space of $\mathbb{R}$, and an open interval subset $I$ of $\mathbb{R}$. If $H = I$, then $H$ is open.

PROOF: For every point $x$ of the real normed space of $\mathbb{R}$ such that $x \in H$ there exists a neighbourhood $N$ of $x$ such that $N \subseteq H$ by [6, (18)], [18, (4)]. □

(17) Let us consider a real number $r$, a set $X$, and a partial function $g$ from $X$ to $\mathbb{R}$. Then $\text{LE-dom}(g, r) = g^{-1}(]-\infty, r[)$.

## 2. Continuity of Two-variable Functions

Now we state the propositions:

(18) Let us consider closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $I \times J$ and $f = g$. Let us consider a real number $e$. Suppose $0 < e$. Then there exists a real number $r$ such that

(i) $0 < r$, and

(ii) for every real numbers $x_1$, $x_2$, $y_1$, $y_2$ such that $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle \in I \times J$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ holds $|g(\langle x_2, y_2\rangle) - g(\langle x_1, y_1\rangle)| < e$.

The theorem is a consequence of (9) and (10).

(19) Let us consider a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. If $f = g$, then $\|f\| = |g|$.

(20) Let us consider a non empty set $X$, a partial function $g$ from $X$ to $\mathbb{R}$, and a subset $A$ of $X$. Then $|g{\restriction}A| = |g|{\restriction}A$.

Proof: For every object $x$ such that $x \in \operatorname{dom}|g{\restriction}A|$ holds $|g{\restriction}A|(x) = (|g|{\restriction}A)(x)$. □

(21) Let us consider a real normed space $S$, a point $x_0$ of $S$, and partial functions $f$, $g$ from $S$ to the real normed space of $\mathbb{R}$. Suppose $f$ is continuous in $x_0$ and $g = \|f\|$. Then $g$ is continuous in $x_0$.

Proof: For every sequence $s_1$ of $S$ such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} g$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $g_* s_1$ is convergent and $g_{/x_0} = \lim(g_* s_1)$. □

(22) Let us consider a set $X$, a real normed space $S$, and partial functions $f$, $g$ from $S$ to the real normed space of $\mathbb{R}$. Suppose $f$ is continuous on $X$ and $g = \|f\|$. Then $g$ is continuous on $X$. The theorem is a consequence of (21).

(23) Let us consider closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $I \times J$ and $f = g$. Let us consider a real number $e$. Suppose $0 < e$. Then there exists a real number $r$ such that

(i) $0 < r$, and

(ii) for every real numbers $x_1$, $x_2$, $y_1$, $y_2$ such that $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle \in I \times J$ and $|x_2 - x_1| < r$ and $|y_2 - y_1| < r$ holds $||g|(\langle x_2, y_2 \rangle) - |g|(\langle x_1, y_1 \rangle)| < e$.

The theorem is a consequence of (19), (22), and (18).

(24) Let us consider a real number $r$, a real normed space $S$, a subset $E$ of $S$, and a partial function $f$ from $S$ to the real normed space of $\mathbb{R}$. Suppose $f$ is continuous on $E$ and $\operatorname{dom} f = E$. Then there exists a subset $H$ of $S$ such that

(i) $H \cap E = f^{-1}(]-\infty, r[)$, and

(ii) $H$ is open.

Proof: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point $t$ of $S$ and there exists a real number $s$ such that $t = \$_1$ and $s = \$_2$ and $0 < s$ and for every object $t_1$ such that $t_1 \in E \cap \{t_1$, where $t_1$ is a point of $S : \|t_1 - t\| < s\}$ holds $f(t_1) \in\ ]-\infty, r[$.

For every object $z$ such that $z \in f^{-1}(]-\infty, r[)$ there exists an object $y$ such that $y \in \mathbb{R}$ and $\mathcal{P}[z, y]$. Consider $R$ being a function from $f^{-1}(]-\infty, r[)$ into $\mathbb{R}$ such that for every object $x$ such that $x \in f^{-1}(]-\infty, r[)$ holds $\mathcal{P}[x, R(x)]$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point $t$ of $S$ such that $t = \$_1$ and $0 < R(\$_1)$ and $\$_2 = \{t_1$, where $t_1$ is a point of $S : \|t_1 - t\| < R(\$_1)\}$. For every object $z$ such that $z \in f^{-1}(]-\infty, r[)$ there exists an object $y$ such that $y \in 2^{\alpha}$ and $\mathcal{Q}[z, y]$, where $\alpha$ is the carrier of $S$.

Consider $B$ being a function from $f^{-1}(]-\infty, r[)$ into $2^{(\text{the carrier of } S)}$ such that for every object $x$ such that $x \in f^{-1}(]-\infty, r[)$ holds $\mathcal{Q}[x, B(x)]$. Set $H = \bigcup \operatorname{rng} B$. For every object $z$, $z \in H \cap E$ iff $z \in f^{-1}(]-\infty, r[)$. For every point $z$ of $S$ such that $z \in H$ there exists a neighbourhood $N$ of $z$ such that $N \subseteq H$. □

## 3. Properties of Projective Functions

Now we state the propositions:

(25) Let us consider non empty sets $X$, $Y$, $Z$, a subset $A$ of $X$, a subset $B$ of $Y$, an element $x$ of $X$, and a partial function $f$ from $X \times Y$ to $Z$. Suppose $\operatorname{dom} f = A \times B$. Then

(i) if $x \in A$, then $\operatorname{dom}(\text{ProjPMap1}(f, x)) = B$, and

(ii) if $x \notin A$, then $\operatorname{dom}(\text{ProjPMap1}(f, x)) = \emptyset$.

(26) Let us consider non empty sets $X$, $Y$, $Z$, a subset $A$ of $X$, a subset $B$ of $Y$, an element $y$ of $Y$, and a partial function $f$ from $X \times Y$ to $Z$. Suppose $\operatorname{dom} f = A \times B$. Then

(i) if $y \in B$, then $\operatorname{dom}(\text{ProjPMap2}(f, y)) = A$, and

(ii) if $y \notin B$, then $\operatorname{dom}(\text{ProjPMap2}(f, y)) = \emptyset$.

(27) Let us consider non empty sets $X$, $Y$, a subset $A$ of $X$, a subset $B$ of $Y$, an element $x$ of $X$, and a partial function $f$ from $X \times Y$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = A \times B$. Then

(i) if $x \in A$, then $\operatorname{dom}(\text{ProjPMap1}(\overline{\mathbb{R}}(f), x)) = B$ and $\operatorname{dom}(\text{ProjPMap1}(|\overline{\mathbb{R}}(f)|, x)) = B$, and

(ii) if $x \notin A$, then $\operatorname{dom}(\text{ProjPMap1}(\overline{\mathbb{R}}(f), x)) = \emptyset$ and $\operatorname{dom}(\text{ProjPMap1}(|\overline{\mathbb{R}}(f)|, x)) = \emptyset$.

The theorem is a consequence of (25).

(28) Let us consider non empty sets $X$, $Y$, a subset $A$ of $X$, a subset $B$ of $Y$, an element $y$ of $Y$, and a partial function $f$ from $X \times Y$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = A \times B$. Then

(i) if $y \in B$, then $\mathrm{dom}(\mathrm{ProjPMap2}(\overline{\mathbb{R}}(f), y)) = A$ and $\mathrm{dom}(\mathrm{ProjPMap2}(|\overline{\mathbb{R}}(f)|, y)) = A$, and

(ii) if $y \notin B$, then $\mathrm{dom}(\mathrm{ProjPMap2}(\overline{\mathbb{R}}(f), y)) = \emptyset$ and $\mathrm{dom}(\mathrm{ProjPMap2}(|\overline{\mathbb{R}}(f)|, y)) = \emptyset$.

The theorem is a consequence of (26).

(29) Let us consider non empty sets $X$, $Y$, a set $Z$, a partial function $f$ from $X \times Y$ to $Z$, an element $x$ of $X$, and an element $y$ of $Y$. Then

(i) $\mathrm{rng}\,\mathrm{ProjPMap1}(f, x) \subseteq \mathrm{rng}\, f$, and

(ii) $\mathrm{rng}\,\mathrm{ProjPMap2}(f, y) \subseteq \mathrm{rng}\, f$.

Let us consider non empty sets $X$, $Y$, a partial function $f$ from $X \times Y$ to $\mathbb{R}$, an element $x$ of $X$, and an element $y$ of $Y$. Now we state the propositions:

(30) (i) $\mathrm{ProjPMap1}(\overline{\mathbb{R}}(f), x)$ is a partial function from $Y$ to $\mathbb{R}$, and

(ii) $\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(f)|, x)$ is a partial function from $Y$ to $\mathbb{R}$, and

(iii) $\mathrm{ProjPMap2}(\overline{\mathbb{R}}(f), y)$ is a partial function from $X$ to $\mathbb{R}$, and

(iv) $\mathrm{ProjPMap2}(|\overline{\mathbb{R}}(f)|, y)$ is a partial function from $X$ to $\mathbb{R}$.

The theorem is a consequence of (29).

(31) (i) $\mathrm{ProjPMap1}(\overline{\mathbb{R}}(f), x) = \overline{\mathbb{R}}(\mathrm{ProjPMap1}(f, x))$, and

(ii) $\mathrm{ProjPMap1}(|\overline{\mathbb{R}}(f)|, x) = |\overline{\mathbb{R}}(\mathrm{ProjPMap1}(f, x))|$, and

(iii) $\mathrm{ProjPMap2}(\overline{\mathbb{R}}(f), y) = \overline{\mathbb{R}}(\mathrm{ProjPMap2}(f, y))$, and

(iv) $\mathrm{ProjPMap2}(|\overline{\mathbb{R}}(f)|, y) = |\overline{\mathbb{R}}(\mathrm{ProjPMap2}(f, y))|$.

(32) (i) $\mathrm{ProjPMap1}(|f|, x) = |\mathrm{ProjPMap1}(f, x)|$, and

(ii) $\mathrm{ProjPMap2}(|f|, y) = |\mathrm{ProjPMap2}(f, y)|$.

Let us consider a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and an element $t$ of $\mathbb{R}$. Now we state the propositions:

(33) If $f$ is continuous on $\mathrm{dom}\, f$ and $f = g$, then $\mathrm{ProjPMap1}(g, t)$ is continuous and $\mathrm{ProjPMap2}(g, t)$ is continuous.

PROOF: For every real number $y_0$ such that $y_0 \in \mathrm{dom}(\mathrm{ProjPMap1}(g, t))$ holds $\mathrm{ProjPMap1}(g, t)$ is continuous in $y_0$. For every real number $x_0$ such that $x_0 \in \mathrm{dom}(\mathrm{ProjPMap2}(g, t))$ holds $\mathrm{ProjPMap2}(g, t)$ is continuous in $x_0$. $\square$

(34) Suppose $f$ is continuous on $\mathrm{dom}\, f$ and $f = g$. Then

(i) $\mathrm{ProjPMap1}(|g|, t)$ is continuous, and

(ii) $\mathrm{ProjPMap2}(|g|, t)$ is continuous.

The theorem is a consequence of (33) and (32).

(35) Suppose $f$ is uniformly continuous on $\operatorname{dom} f$ and $f = g$. Then

(i) ProjPMap1$(g, t)$ is uniformly continuous, and

(ii) ProjPMap2$(g, t)$ is uniformly continuous.

PROOF: For every real number $r$ such that $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real numbers $y_1$, $y_2$ such that $y_1$, $y_2 \in$ dom(ProjPMap1$(g, t)$) and $|y_1 - y_2| < s$ holds $|(\text{ProjPMap1}(g, t))(y_1) - (\text{ProjPMap1}(g, t))(y_2)| < r$. For every real number $r$ such that $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real numbers $x_1$, $x_2$ such that $x_1$, $x_2 \in$ dom(ProjPMap2$(g, t)$) and $|x_1 - x_2| < s$ holds $|(\text{ProjPMap2}(g, t))(x_1) - (\text{ProjPMap2}(g, t))(x_2)| < r$ by [17, (1)]. □

(36) Let us consider an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $\operatorname{dom} f$ and $f = g$ and $P_1 =$ ProjPMap1$(\overline{\mathbb{R}}(g), x)$. Then $P_1$ is continuous. The theorem is a consequence of (31) and (33).

(37) Let us consider an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_2$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $\operatorname{dom} f$ and $f = g$ and $P_2 =$ ProjPMap2$(\overline{\mathbb{R}}(g), y)$. Then $P_2$ is continuous. The theorem is a consequence of (31) and (33).

(38) Let us consider an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $\operatorname{dom} f$ and $f = g$ and $P_1 =$ ProjPMap1$(|\overline{\mathbb{R}}(g)|, x)$. Then $P_1$ is continuous. The theorem is a consequence of (31), (32), and (34).

(39) Let us consider an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $p_2$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f$ is continuous on $\operatorname{dom} f$ and $f = g$ and $p_2 =$ ProjPMap2$(|\overline{\mathbb{R}}(g)|, y)$. Then $p_2$ is continuous. The theorem is a consequence of (31), (32), and (34).

## 4. Integral of Continuous Functions of Two Variables

Let us consider a subset $I$ of $\mathbb{R}$, a non empty, closed interval subset $J$ of $\mathbb{R}$, an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(40) Suppose $x \in I$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_1 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then

(i) $P_1 \restriction J$ is bounded, and

(ii) $P_1$ is integrable on $J$.

The theorem is a consequence of (31), (27), and (33).

(41) Suppose $x \in I$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_1 = \text{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then

(i) $P_1$ is integrable on L-Meas, and

(ii) $\int\limits_J P_1(x)dx = \int P_1 \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_J P_1(x)dx = \int \text{ProjPMap1}(\overline{\mathbb{R}}(g), x) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(x)$.

The theorem is a consequence of (27) and (40).

Let us consider a non empty, closed interval subset $I$ of $\mathbb{R}$, a subset $J$ of $\mathbb{R}$, an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_2$ from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(42) Suppose $y \in J$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_2 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), y)$. Then

(i) $P_2 \restriction I$ is bounded, and

(ii) $P_2$ is integrable on $I$.

The theorem is a consequence of (31), (28), and (33).

(43) Suppose $y \in J$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_2 = \text{ProjPMap2}(\overline{\mathbb{R}}(g), y)$. Then

(i) $P_2$ is integrable on L-Meas, and

(ii) $\int_I P_2(x)dx = \int P_2 \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int_I P_2(x)dx = \int \text{ProjPMap2}(\overline{\mathbb{R}}(g), y) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int_I P_2(x)dx = (\text{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(y)$.

The theorem is a consequence of (28) and (42).

(44) Let us consider a subset $I$ of $\mathbb{R}$, a non empty, closed interval subset $J$ of $\mathbb{R}$, an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_1 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then

(i) $P_1 \restriction J$ is bounded, and

(ii) $P_1$ is integrable on $J$.

The theorem is a consequence of (27) and (38).

(45) Let us consider a subset $I$ of $\mathbb{R}$, a non empty, closed interval subset $J$ of $\mathbb{R}$, an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$, and an element $E$ of L-Field. Suppose $x \in I$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_1 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$ and $E = J$. Then $P_1$ is $E$-measurable. The theorem is a consequence of (27) and (44).

(46) Let us consider a subset $I$ of $\mathbb{R}$, a non empty, closed interval subset $J$ of $\mathbb{R}$, an element $x$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $P_1$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_1 = \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then

(i) $P_1$ is integrable on L-Meas, and

(ii) $\int_J P_1(x)dx = \int P_1 \, \mathrm{d}\,\text{L-Meas}$, and

(iii) $\int_J P_1(x)dx = \int \text{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x) \, \mathrm{d}\,\text{L-Meas}$, and

(iv) $\int_J P_1(x)dx = (\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x)$.

The theorem is a consequence of (27) and (44).

(47) Let us consider a non empty, closed interval subset $I$ of $\mathbb{R}$, a subset $J$ of $\mathbb{R}$, an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, and a partial function $P_2$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)$. Then

(i) $P_2 \restriction I$ is bounded, and

(ii) $P_2$ is integrable on $I$.

The theorem is a consequence of (28) and (39).

(48) Let us consider a non empty, closed interval subset $I$ of $\mathbb{R}$, a subset $J$ of $\mathbb{R}$, an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, a partial function $P_2$ from $\mathbb{R}$ to $\mathbb{R}$, and an element $E$ of L-Field. Suppose $y \in J$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)$ and $E = I$. Then $P_2$ is $E$-measurable. The theorem is a consequence of (28) and (47).

(49) Let us consider a non empty, closed interval subset $I$ of $\mathbb{R}$, a subset $J$ of $\mathbb{R}$, an element $y$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$, and a partial function $P_2$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $y \in J$ and $\operatorname{dom} f = I \times J$ and $f$ is continuous on $I \times J$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)$. Then

(i) $P_2$ is integrable on L-Meas, and

(ii) $\int\limits_I P_2(x)dx = \int P_2 \,\mathrm{d}\,\text{L-Meas}$, and

(iii) $\int\limits_I P_2(x)dx = \int \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y) \,\mathrm{d}\,\text{L-Meas}$, and

(iv) $\int\limits_I P_2(x)dx = (\mathrm{Integral1}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(y)$.

The theorem is a consequence of (28) and (47).

(50) Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and an element $E$ of $\sigma(\mathrm{MeasRect}(\text{L-Field}, \text{L-Field}))$. Suppose $I \times$

$J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$ and $E = I \times J$. Then $g$ is $E$-measurable. The theorem is a consequence of (17), (24), and (15).

(51) Let us consider a subset $I$ of $\mathbb{R}$, a non empty, closed interval subset $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$. Then

(i) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction I$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(ii) Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction I$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (30), (46), and (41).

Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $G_2$ from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(52) Suppose $I \times J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_2 =$ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction I$. Then $G_2$ is continuous.

PROOF: Consider $c$, $d$ being real numbers such that $J = [c, d]$. For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every real numbers $x_1$, $x_2$ such that $|x_2 - x_1| < r$ and $x_1, x_2 \in I$ for every real number $y$ such that $y \in J$ holds $||g|(\langle x_2, y\rangle) - |g|(\langle x_1, y\rangle)| < e$. Set $R = \overline{\mathbb{R}}(g)$. For every elements $x$, $y$ of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $(\mathrm{ProjPMap1}(|R|, x))(y) = |R|(x, y)$ and $|R|(x, y) = |g(\langle x, y\rangle)|$ and $|R|(x, y) = |g|(\langle x, y\rangle)$.

For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every elements $x_1$, $x_2$ of $\mathbb{R}$ such that $|x_2 - x_1| < r$ and $x_1, x_2 \in I$ for every element $y$ of $\mathbb{R}$ such that $y \in J$ holds $|(\mathrm{ProjPMap1}(|R|, x_2))(y) - (\mathrm{ProjPMap1}(|R|, x_1))(y)| < e$. For every real numbers $x_0$, $r$ such that $x_0 \in I$ and $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real number $x_1$ such that $x_1 \in I$ and $|x_1 - x_0| < s$ holds $|G_2(x_1) - G_2(x_0)| < r$. □

(53) Suppose $I \times J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_2 =$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction I$. Then $G_2$ is continuous.

PROOF: Consider $c$, $d$ being real numbers such that $J = [c, d]$. For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every real numbers $x_1$, $x_2$ such that $|x_2 - x_1| < r$ and $x_1, x_2 \in I$ for every real number $y$ such that $y \in J$ holds $|g(\langle x_2, y\rangle) - g(\langle x_1, y\rangle)| < e$. Set $R = \overline{\mathbb{R}}(g)$.

For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every elements $x_1$, $x_2$ of $\mathbb{R}$ such that

$|x_2 - x_1| < r$ and $x_1$, $x_2 \in I$ for every element $y$ of $\mathbb{R}$ such that $y \in J$ holds $|(\mathrm{ProjPMap1}(R, x_2))(y) - (\mathrm{ProjPMap1}(R, x_1))(y)| < e$. For every real numbers $x_0$, $r$ such that $x_0 \in I$ and $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real number $x_1$ such that $x_1 \in I$ and $|x_1 - x_0| < s$ holds $|G_2(x_1) - G_2(x_0)| < r$. □

(54) Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $g$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $f$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \mathrm{dom}\, g$ and $g$ is continuous on $I \times J$ and $g = f$. Then

(i) $\mathrm{Integral1}(\text{L-Meas}, |\overline{\mathbb{R}}(f)|) \restriction J$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(ii) $\mathrm{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(f)) \restriction J$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (30), (49), and (43).

Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $G_1$ from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(55) Suppose $I \times J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_1 = \mathrm{Integral1}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|) \restriction J$. Then $G_1$ is continuous.

PROOF: Consider $a$, $b$ being real numbers such that $I = [a, b]$. For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every real numbers $y_1$, $y_2$ such that $|y_2 - y_1| < r$ and $y_1$, $y_2 \in J$ for every real number $x$ such that $x \in I$ holds $||g|(\langle x, y_2 \rangle) - |g|(\langle x, y_1 \rangle)| < e$. Set $R = \overline{\mathbb{R}}(g)$. For every elements $x$, $y$ of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $(\mathrm{ProjPMap2}(|R|, y))(x) = |R|(x, y)$ and $|R|(x, y) = |g(\langle x, y \rangle)|$ and $|R|(x, y) = |g|(\langle x, y \rangle)$.

For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every elements $y_1$, $y_2$ of $\mathbb{R}$ such that $|y_2 - y_1| < r$ and $y_1$, $y_2 \in J$ for every element $x$ of $\mathbb{R}$ such that $x \in I$ holds $|(\mathrm{ProjPMap2}(|R|, y_2))(x) - (\mathrm{ProjPMap2}(|R|, y_1))(x)| < e$. For every real numbers $y_0$, $r$ such that $y_0 \in J$ and $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real number $y_1$ such that $y_1 \in J$ and $|y_1 - y_0| < s$ holds $|G_1(y_1) - G_1(y_0)| < r$. □

(56) Suppose $I \times J = \mathrm{dom}\, f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_1 = \mathrm{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction J$. Then $G_1$ is continuous.

PROOF: Consider $a$, $b$ being real numbers such that $I = [a, b]$. For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every real numbers $y_1$, $y_2$ such that $|y_2 - y_1| < r$ and $y_1$, $y_2 \in J$ for every real number $x$ such that $x \in I$ holds $|g(\langle x, y_2 \rangle) - g(\langle x, y_1 \rangle)| < e$. Set $R = \overline{\mathbb{R}}(g)$.

For every real number $e$ such that $0 < e$ there exists a real number $r$ such that $0 < r$ and for every elements $y_1$, $y_2$ of $\mathbb{R}$ such that $|y_2 - y_1| < r$ and $y_1, y_2 \in J$ for every element $x$ of $\mathbb{R}$ such that $x \in I$ holds $|(\text{ProjPMap2}(R, y_2))(x) - (\text{ProjPMap2}(R, y_1))(x)| < e$. For every real numbers $y_0$, $r$ such that $y_0 \in J$ and $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real number $y_1$ such that $y_1 \in J$ and $|y_1 - y_0| < s$ holds $|G_1(y_1) - G_1(y_0)| < r$. □

(57) Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \operatorname{dom} f$ and $f$ is continuous on $I \times J$ and $f = g$. Then

- (i) $g$ is integrable on ProdMeas(L-Meas, L-Meas), and
- (ii) for every element $x$ of $\mathbb{R}$, $(\text{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x) < +\infty$, and
- (iii) for every element $y$ of $\mathbb{R}$, $(\text{Integral1}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(y) < +\infty$, and
- (iv) for every element $U$ of L-Field, $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is $U$-measurable, and
- (v) for every element $V$ of L-Field, $\text{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is $V$-measurable, and
- (vi) $\text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on L-Meas, and
- (vii) $\text{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is integrable on L-Meas, and
- (viii) $\int g \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \,\mathrm{d}\,\text{L-Meas}$, and
- (ix) $\int g \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int \text{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \,\mathrm{d}\,\text{L-Meas}$.

(58) Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $G_2$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \operatorname{dom} f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_2 = \text{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction I$. Then $\int \overline{\mathbb{R}}(g) \,\mathrm{d}\,\text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int\limits_I G_2(x)dx$.

Proof: Set $R = \overline{\mathbb{R}}(g)$. Set $N_1 = \mathbb{R} \setminus I$. Set $R_2 = \text{Integral2}(\text{L-Meas}, R)$. Set $F_1 = R_2 \restriction N_1$. $G_2$ is continuous. For every element $x$ of $\mathbb{R}$ such that $x \in \operatorname{dom} F_1$ holds $F_1(x) = 0$. □

(59) Let us consider non empty, closed interval subsets $I$, $J$ of $\mathbb{R}$, a partial function $f$ from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function $g$ from $\mathbb{R} \times \mathbb{R}$ to

$\mathbb{R}$, and a partial function $G_1$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \operatorname{dom} f$ and $f$ is continuous on $I \times J$ and $f = g$ and $G_1 = \text{Integral1}(\text{L-Meas}, \overline{\mathbb{R}}(g)) \restriction J$. Then $\int \overline{\mathbb{R}}(g) \, \mathrm{d}\, \text{ProdMeas}(\text{L-Meas}, \text{L-Meas}) = \int\limits_J G_1(x) dx$.

PROOF: Set $R = \overline{\mathbb{R}}(g)$. Set $N_2 = \mathbb{R} \setminus J$. Set $R_1 = \text{Integral1}(\text{L-Meas}, R)$. Set $F_1 = R_1 \restriction N_2$. $G_1 \restriction J$ is bounded and $G_1$ is integrable on $J$. For every element $y$ of $\mathbb{R}$ such that $y \in \operatorname{dom} F_1$ holds $F_1(y) = 0$. □

## References


[1] Tom M. Apostol. *Calculus*, volume II. Wiley, second edition, 1969.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Improving real analysis in Coq: A user-friendly approach to integrals and derivatives. In Chris Hawblitzel and Dale Miller, editors, *Certified Programs and Proofs – Second International Conference, CPP 2012, Kyoto, Japan, December 13–15, 2012. Proceedings*, volume 7679 of *Lecture Notes in Computer Science*, pages 289–304. Springer, 2012. doi:10.1007/978-3-642-35308-6_22.

[4] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Formalization of real analysis: A survey of proof assistants and libraries. *Mathematical Structures in Computer Science*, 26:1196–1233, 2015.

[5] Noboru Endou. Improper integral. Part II. *Formalized Mathematics*, 29(**4**):279–294, 2021. doi:10.2478/forma-2021-0024.

[6] Noboru Endou. Antiderivatives and integration. *Formalized Mathematics*, 31(1):131–141, 2023. doi:10.2478/forma-2023-0012.

[7] Noboru Endou. Product pre-measure. *Formalized Mathematics*, 24(**1**):69–79, 2016. doi:10.1515/forma-2016-0006.

[8] Noboru Endou. Reconstruction of the one-dimensional Lebesgue measure. *Formalized Mathematics*, 28(**1**):93–104, 2020. doi:10.2478/forma-2020-0008.

[9] Noboru Endou. Fubini's theorem. *Formalized Mathematics*, 27(**1**):67–74, 2019. doi:10.2478/forma-2019-0007.

[10] Noboru Endou. Relationship between the Riemann and Lebesgue integrals. *Formalized Mathematics*, 29(**4**):185–199, 2021. doi:10.2478/forma-2021-0018.

[11] Noboru Endou. Absolutely integrable functions. *Formalized Mathematics*, 30(**1**):31–51, 2022. doi:10.2478/forma-2022-0004.

[12] Jacques D. Fleuriot. On the mechanization of real analysis in Isabelle/HOL. In Mark Aagaard and John Harrison, editors, *Theorem Proving in Higher Order Logics*, pages 145–161. Springer Berlin Heidelberg, 2000. ISBN 978-3-540-44659-0.

[13] Ruben Gamboa. *Continuity and Differentiability*, pages 301–315. Springer US, 2000. ISBN 978-1-4757-3188-0. doi:10.1007/978-1-4757-3188-0_18.

[14] Adam Grabowski and Christoph Schwarzweller. On duplication in mathematical repositories. In Serge Autexier, Jacques Calmet, David Delahaye, Patrick D. F. Ion, Laurence Rideau, Renaud Rioboo, and Alan P. Sexton, editors, *Intelligent Computer Mathematics, 10th International Conference, AISC 2010, 17th Symposium, Calculemus 2010, and 9th International Conference, MKM 2010, Paris, France, July 5–10, 2010. Proceedings*, volume 6167 of *Lecture Notes in Computer Science*, pages 300–314. Springer, 2010. doi:10.1007/978-3-642-14128-7_26.

[15] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Mizar in a nutshell. *Journal of Formalized Reasoning*, 3(2):153–245, 2010.

[16] Serge Lang. *Calculus of Several Variables*. Springer, third edition, 2012.

[17] Kazuhisa Nakasho, Yuichi Futa, and Yasunari Shidama. Implicit function theorem. Part I. *Formalized Mathematics*, 25(**4**):269–281, 2017. doi:10.1515/forma-2017-0026.

[18] Keiko Narita, Noboru Endou, and Yasunari Shidama. Weak convergence and weak* convergence. *Formalized Mathematics*, 23(**3**):231–241, 2015. doi:10.1515/forma-2015-0019.

[19] Edward S. Smith, Meyer Salkover, and Howard K. Justice. *Calculus*. John Wiley and Sons, second edition, 1958.

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