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Conway Numbers – Formal Introduction

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Summary. Surreal numbers, a fascinating mathematical concept introduced by John Conway, have attracted considerable interest due to their unique properties. In this article, we formalize the basic concept of surreal numbers close to the original Conway's convention in the field of combinatorial game theory. We define surreal numbers with the pre-order in the Mizar system which satisfy the following condition: $x \leqslant y$ iff $L_x \ll \{y\} \wedge \{x\} \ll R_y$.

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Introduction

The surreal numbers have been discovered by J. Conway and they are described in the 0th part of his book [1]. Using a remarkably simple set of rules, he showed that a rich algebraic structure, as totally ordered proper class that form an ordered field could be constructed. However, his construction combines transfinite induction recursion [2] with properties of proper classes, and has been challenged from a formal point of view. We have chosen to construct surreal numbers based on transfinite induction (for recent quite sophisticated use of these second order statements, see [10] and [11]), in contrast to the formalisation in other systems [7], [9].

Imitating the induction recursion in the Mizar system, and, at the same time, to come as close as possible to the Conway convention with a non anti-symmetric pre-order we have extracted an additional fundamental step. We introduce the functor of $\mathrm{Day}_R\alpha$ for a given ordinal α and relation R as well as the properties of the pre-order on a set D which will play the role of the $\mathrm{Day}\alpha$, independently. Then we extract the crucial dependencies between $\mathrm{Day}\alpha$ and the pre-order to remove parameters and finally define the concept of surreal numbers in the Mizar system [6].

The formalization follows [1], [3], [4], [5] and is an independent approach to that introduced by R. Nittka [8].

1. Construction of Games on α-Day

From now on α, α_1, α_2, β, β_1, β_2, γ, θ denote ordinal numbers, R, S denote binary relations, and a, b, c, o, l, r denote objects. Let x be an object. We introduce the notation L_x as a synonym of $(x)_{\mathbf{1}}$ and R_x as a synonym of $(x)_{\mathbf{2}}$.

Note that the functor L_x yields a set. Let us observe that the functor R_x yields a set. Let us consider a and b. Let θ be a set. We say that $a \leqslant_\theta b$ if and only if

(Def. 1) $\langle a, b\rangle \in \theta$.

We introduce the notation $b \succeq_\theta a$ as a synonym of $a \leqslant_\theta b$.

Let L, R be sets. We say that $L \gg_\theta R$ if and only if

(Def. 2) if $l \in L$ and $r \in R$, then $l \succeq_\theta r$.

We say that $L \ll_\theta R$ if and only if

(Def. 3) if $l \in L$ and $r \in R$, then not $l \succeq_\theta r$.

Let us consider α. The functor $\mathrm{Games}(\alpha)$ yielding a set is defined by

(Def. 4) there exists a transfinite sequence L such that $it = L(\alpha)$ and $\mathrm{dom}\, L = \mathrm{succ}\, \alpha$ and for every θ such that $\theta \in \mathrm{succ}\, \alpha$ holds $L(\theta) = 2^{\bigcup \mathrm{rng}(L \restriction \theta)} \times 2^{\bigcup \mathrm{rng}(L \restriction \theta)}$.

Let us note that $\mathrm{Games}(\alpha)$ is non empty and relation-like. Now we state the propositions:

(1) If $\alpha \subseteq \beta$, then $\mathrm{Games}(\alpha) \subseteq \mathrm{Games}(\beta)$.

Proof: Consider L_1 being a transfinite sequence such that $\mathrm{Games}(\alpha) = L_1(\alpha)$ and $\mathrm{dom}\, L_1 = \mathrm{succ}\, \alpha$ and for every ordinal number θ such that $\theta \in \mathrm{succ}\, \alpha$ holds $L_1(\theta) = 2^{\bigcup \mathrm{rng}(L_1 \restriction \theta)} \times 2^{\bigcup \mathrm{rng}(L_1 \restriction \theta)}$. Consider L_2 being a transfinite sequence such that $\mathrm{Games}(\beta) = L_2(\beta)$ and $\mathrm{dom}\, L_2 = \mathrm{succ}\, \beta$ and for every ordinal number θ such that $\theta \in \mathrm{succ}\, \beta$ holds $L_2(\theta) = 2^{\bigcup \mathrm{rng}(L_2 \restriction \theta)} \times 2^{\bigcup \mathrm{rng}(L_2 \restriction \theta)}$.

Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \subseteq \alpha$, then $L_1(\$_1) = L_2(\$_1)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\mathrm{rng}(L_1 \restriction \alpha) \subseteq \mathrm{rng}(L_2 \restriction \beta)$. □

(2) $\mathrm{Games}(0) = \{\langle \emptyset, \emptyset \rangle\}$.

(3) Let us consider a transfinite sequence L, and θ. Suppose $\mathrm{dom}\, L = \mathrm{succ}\, \theta$ and for every α such that $\alpha \in \mathrm{succ}\, \theta$ holds $L(\alpha) = 2^{\bigcup \mathrm{rng}(L \restriction \alpha)} \times 2^{\bigcup \mathrm{rng}(L \restriction \alpha)}$. If $\alpha \in \mathrm{succ}\, \theta$, then $L(\alpha) = \mathrm{Games}(\alpha)$.

PROOF: Consider L_0 being a transfinite sequence such that $\mathrm{Games}(\theta) = L_0(\theta)$ and $\mathrm{dom}\, L_0 = \mathrm{succ}\, \theta$ and for every ordinal number α such that $\alpha \in \mathrm{succ}\, \theta$ holds $L_0(\alpha) = 2^{\bigcup \mathrm{rng}(L_0 \restriction \alpha)} \times 2^{\bigcup \mathrm{rng}(L_0 \restriction \alpha)}$. Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \theta$, then $L_0(\$_1) = L(\$_1)$.

For every ordinal number α such that for every ordinal number γ such that $\gamma \in \alpha$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\alpha]$. For every ordinal number α, $\mathcal{P}[\alpha]$. □

(4) $o \in \mathrm{Games}(\theta)$ if and only if o is pair and for every a such that $a \in \mathrm{L}_o \cup \mathrm{R}_o$ there exists α such that $\alpha \in \theta$ and $a \in \mathrm{Games}(\alpha)$.

PROOF: Consider L being a transfinite sequence such that $\mathrm{Games}(\theta) = L(\theta)$ and $\mathrm{dom}\, L = \mathrm{succ}\, \theta$ and for every α such that $\alpha \in \mathrm{succ}\, \theta$ holds $L(\alpha) = 2^{\bigcup \mathrm{rng}(L \restriction \alpha)} \times 2^{\bigcup \mathrm{rng}(L \restriction \alpha)}$. If $o \in \mathrm{Games}(\theta)$, then o is pair and for every object x such that $x \in \mathrm{L}_o \cup \mathrm{R}_o$ there exists an ordinal number β such that $\beta \in \theta$ and $x \in \mathrm{Games}(\beta)$. $\mathrm{L}_o \cup \mathrm{R}_o \subseteq \bigcup \mathrm{rng}(L \restriction \theta)$. □

Let us consider α. The functor $\mathrm{BeforeGames}(\alpha)$ yielding a subset of $\mathrm{Games}(\alpha)$ is defined by

(Def. 5) $a \in \mathit{it}$ iff there exists θ such that $\theta \in \alpha$ and $a \in \mathrm{Games}(\theta)$.

Now we state the proposition:

(5) If $\alpha \subseteq \beta$, then $\mathrm{BeforeGames}(\alpha) \subseteq \mathrm{BeforeGames}(\beta)$.

Let us consider θ and R. The functor $\mathrm{Day}_R \theta$ yielding a subset of $\mathrm{Games}(\theta)$ is defined by

(Def. 6) there exists a transfinite sequence L such that $\mathit{it} = L(\theta)$ and $\mathrm{dom}\, L = \mathrm{succ}\, \theta$ and for every α such that $\alpha \in \mathrm{succ}\, \theta$ holds $L(\alpha) = \{x$, where x is an element of $\mathrm{Games}(\alpha) : \mathrm{L}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{R}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{L}_x \ll_R \mathrm{R}_x\}$.

2. CONSTRUCTION OF PREORDER ON THE α-DAY

Let us consider R. We say that R is almost **No** order if and only if

(Def. 7) there exists θ such that $R \subseteq \mathrm{Day}_R \theta \times \mathrm{Day}_R \theta$.

Now we state the propositions:

(6) Let us consider a transfinite sequence L. Suppose $\mathrm{dom}\, L = \mathrm{succ}\, \theta$ and for every α such that $\alpha \in \mathrm{succ}\, \theta$ holds $L(\alpha) = \{x$, where x is an element of $\mathrm{Games}(\alpha) : \mathrm{L}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{R}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{L}_x \ll_R \mathrm{R}_x\}$. If $\alpha \in \mathrm{succ}\, \theta$, then $L(\alpha) = \mathrm{Day}_R \alpha$.

Proof: Consider L_0 being a transfinite sequence such that $\mathrm{Day}_R\delta = L_0(\delta)$ and $\mathrm{dom}\, L_0 = \mathrm{succ}\,\delta$ and for every ordinal number α such that $\alpha \in \mathrm{succ}\,\delta$ holds $L_0(\alpha) = \{x$, where x is an element of $\mathrm{Games}(\alpha) : \mathrm{L}_x \subseteq \bigcup \mathrm{rng}(L_0 \restriction \alpha)$ and $\mathrm{R}_x \subseteq \bigcup \mathrm{rng}(L_0 \restriction \alpha)$ and $\mathrm{L}_x \ll_R \mathrm{R}_x\}$.

Define $\mathcal{P}$[ordinal number] $\equiv$ if $\$_1 \subseteq \delta$, then $L_0(\$_1) = L(\$_1)$. For every ordinal number α such that for every ordinal number γ such that $\gamma \in \alpha$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\alpha]$. For every α, $\mathcal{P}[\alpha]$. $\square$

(7) Let us consider an element x of $\mathrm{Games}(\theta)$. Then $x \in \mathrm{Day}_R\theta$ if and only if $\mathrm{L}_x \ll_R \mathrm{R}_x$ and for every o such that $o \in \mathrm{L}_x \cup \mathrm{R}_x$ there exists α such that $\alpha \in \theta$ and $o \in \mathrm{Day}_R\alpha$.

Proof: Consider L being a transfinite sequence such that $\mathrm{Day}_R\theta = L(\theta)$ and $\mathrm{dom}\, L = \mathrm{succ}\,\theta$ and for every α such that $\alpha \in \mathrm{succ}\,\theta$ holds $L(\alpha) = \{x$, where x is an element of $\mathrm{Games}(\alpha) : \mathrm{L}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{R}_x \subseteq \bigcup \mathrm{rng}(L \restriction \alpha)$ and $\mathrm{L}_x \ll_R \mathrm{R}_x\}$. If $\alpha \in \mathrm{Day}_R\theta$, then $\mathrm{L}_\alpha \ll_R \mathrm{R}_\alpha$ and for every object x such that $x \in \mathrm{L}_\alpha \cup \mathrm{R}_\alpha$ there exists an ordinal number β such that $\beta \in \theta$ and $x \in \mathrm{Day}_R\beta$. $\mathrm{L}_\alpha \cup \mathrm{R}_\alpha \subseteq \bigcup \mathrm{rng}(L \restriction \theta)$. $\square$

(8) $\mathrm{Day}_R 0 = \mathrm{Games}(0)$. The theorem is a consequence of (2) and (7).

(9) If $\alpha \subseteq \beta$, then $\mathrm{Day}_R\alpha \subseteq \mathrm{Day}_R\beta$. The theorem is a consequence of (7) and (1).

Let us consider R and α. Let us note that $\mathrm{Day}_R\alpha$ is non empty. Now we state the proposition:

(10) Suppose $\beta \subseteq \alpha$ and $R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha)) = S \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. Then $\mathrm{Day}_R\beta = \mathrm{Day}_S\beta$. The theorem is a consequence of (5).

Let us consider R and o. Assume there exists θ such that $o \in \mathrm{Day}_R\theta$. The functor $\mathfrak{born}_R o$ yielding an ordinal number is defined by

(Def. 8) $o \in \mathrm{Day}_R it$ and for every θ such that $o \in \mathrm{Day}_R\theta$ holds $it \subseteq \theta$.

Now we state the propositions:

(11) Suppose $R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha)) = S \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. If $a \in \mathrm{Day}_R\alpha$, then $\mathfrak{born}_R a = \mathfrak{born}_S a$. The theorem is a consequence of (10).

(12) If $o \in \mathrm{Games}(\theta)$ and $o \notin \mathrm{Day}_R\theta$, then $o \notin \mathrm{Day}_R\alpha$.

Proof: Define $\mathcal{P}$[ordinal number] $\equiv$ for every object x for every ordinal number θ such that $x \in (\mathrm{Games}(\theta)) \setminus (\mathrm{Day}_R\theta)$ holds $x \notin \mathrm{Day}_R\$_1$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $\square$

Let us consider R, α, and β. The functor $\mathrm{OpenProd}_R(\alpha, \beta)$ yielding a binary relation on $\mathrm{Day}_R\alpha$ is defined by

(Def. 9) for every elements x, y of $\mathrm{Day}_R\alpha$, $\langle x, y\rangle \in it$ iff $\mathfrak{born}_R x, \mathfrak{born}_R y \in \alpha$ or $\mathfrak{born}_R x = \alpha$ and $\mathfrak{born}_R y \in \beta$ or $\mathfrak{born}_R x \in \beta$ and $\mathfrak{born}_R y = \alpha$.

The functor $\mathrm{ClosedProd}_R(\alpha, \beta)$ yielding a binary relation on $\mathrm{Day}_R\alpha$ is defined by

(Def. 10) for every elements x, y of $\mathrm{Day}_R\alpha$, $\langle x, y\rangle \in it$ iff $\mathfrak{born}_R x, \mathfrak{born}_R y \in \alpha$ or $\mathfrak{born}_R x = \alpha$ and $\mathfrak{born}_R y \subseteq \beta$ or $\mathfrak{born}_R x \subseteq \beta$ and $\mathfrak{born}_R y = \alpha$.

Now we state the propositions:

(13) Suppose $\alpha_1 \in \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 \subseteq \beta_2$. Then $\mathrm{OpenProd}_R(\alpha_1, \beta_1) \subseteq \mathrm{OpenProd}_R(\alpha_2, \beta_2)$. The theorem is a consequence of (9).

(14) Suppose $R\cap(\mathrm{BeforeGames}(\alpha)\times\mathrm{BeforeGames}(\alpha)) = S\cap(\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. Then $\mathrm{OpenProd}_R(\alpha, \beta) = \mathrm{OpenProd}_S(\alpha, \beta)$.
PROOF: $\mathrm{Day}_R\alpha = \mathrm{Day}_S\alpha$. If $\langle x, y\rangle \in \mathrm{OpenProd}_R(\alpha, \beta)$, then $\langle x, y\rangle \in \mathrm{OpenProd}_S(\alpha, \beta)$. $\mathfrak{born}_R x = \mathfrak{born}_S x$ and $\mathfrak{born}_R y = \mathfrak{born}_S y$. □

(15) Suppose $R\cap(\mathrm{BeforeGames}(\alpha)\times\mathrm{BeforeGames}(\alpha)) = S\cap(\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. Then $\mathrm{ClosedProd}_R(\alpha, \beta) = \mathrm{ClosedProd}_S(\alpha, \beta)$.
PROOF: $\mathrm{Day}_R\alpha = \mathrm{Day}_S\alpha$. If $\langle x, y\rangle \in \mathrm{ClosedProd}_R(\alpha, \beta)$, then $\langle x, y\rangle \in \mathrm{ClosedProd}_S(\alpha, \beta)$. $\mathfrak{born}_R x = \mathfrak{born}_S x$ and $\mathfrak{born}_R y = \mathfrak{born}_S y$. □

(16) $\mathrm{OpenProd}_R(\alpha, \beta) \subseteq \mathrm{ClosedProd}_R(\alpha, \beta)$.

(17) Suppose $\alpha_1 \in \alpha_2$ or $\alpha_1 = \alpha_2$ and $\beta_1 \subseteq \beta_2$. Then $\mathrm{ClosedProd}_R(\alpha_1, \beta_1) \subseteq \mathrm{ClosedProd}_R(\alpha_2, \beta_2)$. The theorem is a consequence of (9).

(18) If $\beta \in \gamma$, then $\mathrm{ClosedProd}_R(\alpha, \beta) \subseteq \mathrm{OpenProd}_R(\alpha, \gamma)$.

(19) If $\alpha \in \beta$, then $\mathrm{ClosedProd}_R(\alpha, \beta) \subseteq \mathrm{OpenProd}_R(\alpha, \beta)$.

Let X, R be sets. We say that R preserves **No** comparison on X if and only if

(Def. 11) for every objects a, b such that $\langle a, b\rangle \in X$ holds $a \leqslant_R b$ iff $\mathrm{L}_a \ll_R \{b\}$ and $\{a\} \ll_R \mathrm{R}_b$.

Now we state the propositions:

(20) Suppose R is almost **No** order and S is almost **No** order and $R \cap \mathrm{OpenProd}_R(\alpha, \beta) = S \cap \mathrm{OpenProd}_S(\alpha, \beta)$. Then $R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha)) = S \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$.
PROOF: Consider R_0 being an ordinal number such that $R \subseteq \mathrm{Day}_R R_0 \times \mathrm{Day}_R R_0$. Consider S_0 being an ordinal number such that $S \subseteq \mathrm{Day}_S S_0 \times \mathrm{Day}_S S_0$. If $\langle y, z\rangle \in R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$, then $\langle y, z\rangle \in S \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$.

Consider A_4 being an ordinal number such that $A_4 \in \alpha$ and $y \in \mathrm{Games}(A_4)$. Consider A_5 being an ordinal number such that $A_5 \in \alpha$ and $z \in \mathrm{Games}(A_5)$. $\mathrm{Day}_S A_4 \subseteq \mathrm{Day}_S\alpha$ and $\mathrm{Day}_S A_5 \subseteq \mathrm{Day}_S\alpha$. $y \in \mathrm{Day}_S A_4$ and $z \in \mathrm{Day}_S A_5$. □

(21) Suppose R is almost **No** order and S is almost **No** order and $R \cap \mathrm{OpenProd}_R(\alpha, \beta) = S \cap \mathrm{OpenProd}_S(\alpha, \beta)$ and R preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha, \beta)$ and S preserves **No** comparison on $\mathrm{ClosedProd}_S(\alpha, \beta)$. Then $R \cap \mathrm{ClosedProd}_R(\alpha, \beta) = S \cap \mathrm{ClosedProd}_S(\alpha, \beta)$. The theorem is a consequence of (16) and (19).

(22) Suppose R is almost **No** order and S is almost **No** order and $R \cap \mathrm{OpenProd}_R(\alpha, 0) = S \cap \mathrm{OpenProd}_S(\alpha, 0)$ and R preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha, \beta)$ and S preserves **No** comparison on $\mathrm{ClosedProd}_S(\alpha, \beta)$. Then $R \cap \mathrm{ClosedProd}_R(\alpha, \beta) = S \cap \mathrm{ClosedProd}_S(\alpha, \beta)$.
Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \subseteq \beta$, then $R \cap \mathrm{ClosedProd}_R(\alpha, \$_1) = S \cap \mathrm{ClosedProd}_S(\alpha, \$_1)$. $R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha)) = S \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(23) Suppose R is almost **No** order and S is almost **No** order and R preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha, \beta)$ and S preserves **No** comparison on $\mathrm{ClosedProd}_S(\alpha, \beta)$. Then $R \cap \mathrm{ClosedProd}_R(\alpha, \beta) = S \cap \mathrm{ClosedProd}_S(\alpha, \beta)$.
Proof: Define $\mathcal{P}[\text{ordinal number}] \equiv$ if $\$_1 \in \alpha$, then $R \cap \mathrm{ClosedProd}_R(\$_1, \$_1) = S \cap \mathrm{ClosedProd}_S(\$_1, \$_1)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. $R \cap \mathrm{OpenProd}_R(\alpha, 0) \subseteq S \cap \mathrm{OpenProd}_S(\alpha, 0)$. $S \cap \mathrm{OpenProd}_S(\alpha, 0) \subseteq R \cap \mathrm{OpenProd}_R(\alpha, 0)$. □

(24) Let us consider transfinite sequences L_3, L_4. Suppose $\mathrm{dom}\, L_3 = \mathrm{dom}\, L_4$ and for every α such that $\alpha \in \mathrm{dom}\, L_3$ holds there exist ordinal numbers a, b and there exists a binary relation R such that $R = L_4(\alpha)$ and $L_3(\alpha) = \mathrm{ClosedProd}_R(a, b)$ and $L_4(\alpha)$ is a binary relation and for every binary relation R such that $R = L_4(\alpha)$ holds R preserves **No** comparison on $L_3(\alpha)$ and $R \subseteq L_3(\alpha)$. Then

(i) $\bigcup \mathrm{rng}\, L_4$ is a binary relation, and

(ii) for every R such that $R = \bigcup \mathrm{rng}\, L_4$ holds R preserves **No** comparison on $\bigcup \mathrm{rng}\, L_3$ and $R \subseteq \bigcup \mathrm{rng}\, L_3$ and for every ordinal numbers α, a, b and for every S such that $\alpha \in \mathrm{dom}\, L_3$ and $S = L_4(\alpha)$ and $L_3(\alpha) = \mathrm{ClosedProd}_S(a, b)$ holds $R \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a)) = S \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a))$.

Proof: $\bigcup \mathrm{rng}\, L_4$ is relation-like. $R \subseteq \bigcup \mathrm{rng}\, L_3$. R preserves **No** comparison on $\bigcup \mathrm{rng}\, L_3$. $R \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a)) \subseteq S \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a))$. $S \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a)) \subseteq R \cap (\mathrm{BeforeGames}(a) \times \mathrm{BeforeGames}(a))$. □

(25) $\langle a, b\rangle \in (\mathrm{ClosedProd}_R(\alpha,\beta)) \setminus (\mathrm{OpenProd}_R(\alpha,\beta))$ if and only if $a, b \in \mathrm{Day}_R\alpha$ and ($\mathfrak{born}_R a = \alpha$ and $\mathfrak{born}_R b = \beta$ or $\mathfrak{born}_R a = \beta$ and $\mathfrak{born}_R b = \alpha$).
PROOF: If $\langle a, b\rangle \in (\mathrm{ClosedProd}_R(\alpha,\beta)) \setminus (\mathrm{OpenProd}_R(\alpha,\beta))$, then $a, b \in \mathrm{Day}_R\alpha$ and ($\mathfrak{born}_R a = \alpha$ and $\mathfrak{born}_R b = \beta$ or $\mathfrak{born}_R a = \beta$ and $\mathfrak{born}_R b = \alpha$). $\langle a, b\rangle \notin \mathrm{OpenProd}_R(\alpha,\beta)$. □

(26) Suppose R preserves **No** comparison on $\mathrm{OpenProd}_R(\alpha,\beta)$ and $R \subseteq \mathrm{OpenProd}_R(\alpha,\beta)$. Then there exists S such that

(i) $R \subseteq S$, and

(ii) S preserves **No** comparison on $\mathrm{ClosedProd}_S(\alpha,\beta)$, and

(iii) $S \subseteq \mathrm{ClosedProd}_S(\alpha,\beta)$.

PROOF: Set $C_1 = \{\langle x, y\rangle$, where x, y are elements of $\mathrm{Day}_R\alpha$: ($\mathfrak{born}_R x = \beta$ and $\mathfrak{born}_R y = \alpha$ or $\mathfrak{born}_R x = \alpha$ and $\mathfrak{born}_R y = \beta$) and $\mathrm{L}_x \ll_R \{y\}$ and $\{x\} \ll_R \mathrm{R}_y\}$. C_1 is relation-like. Reconsider $R_1 = R \cup C_1$ as a binary relation. $R_1 \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha)) \subseteq R \cap (\mathrm{BeforeGames}(\alpha) \times \mathrm{BeforeGames}(\alpha))$. $R_1 \subseteq \mathrm{ClosedProd}_R(\alpha,\beta)$. R_1 preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha,\beta)$. □

(27) Suppose there exists R such that R preserves **No** comparison on $\mathrm{OpenProd}_R(\alpha,\emptyset)$ and $R \subseteq \mathrm{OpenProd}_R(\alpha,\emptyset)$. Then there exists S such that

(i) S preserves **No** comparison on $\mathrm{ClosedProd}_S(\alpha,\beta)$, and

(ii) $S \subseteq \mathrm{ClosedProd}_S(\alpha,\beta)$.

PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ there exists a binary relation R such that R preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha,\$_1)$ and $R \subseteq \mathrm{ClosedProd}_R(\alpha,\$_1)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(28) There exists R such that

(i) R preserves **No** comparison on $\mathrm{ClosedProd}_R(\alpha,\beta)$, and

(ii) $R \subseteq \mathrm{ClosedProd}_R(\alpha,\beta)$.

PROOF: Define $\mathcal{P}$[ordinal number] $\equiv$ for every ordinal number β, there exists a binary relation R such that R preserves **No** comparison on $\mathrm{ClosedProd}_R(\$_1,\beta)$ and $R \subseteq \mathrm{ClosedProd}_R(\$_1,\beta)$. For every ordinal number δ such that for every ordinal number γ such that $\gamma \in \delta$ holds $\mathcal{P}[\gamma]$ holds $\mathcal{P}[\delta]$. For every ordinal number δ, $\mathcal{P}[\delta]$. □

(29) If $\alpha \in \beta$, then $\mathrm{ClosedProd}_R(\alpha,\alpha) = \mathrm{OpenProd}_R(\alpha,\beta)$.
PROOF: $\mathrm{ClosedProd}_R(\alpha,\alpha) \subseteq \mathrm{ClosedProd}_R(\alpha,\beta)$. $\mathrm{ClosedProd}_R(\alpha,\beta) \subseteq \mathrm{ClosedProd}_R(\alpha,\alpha)$. $\mathrm{ClosedProd}_R(\alpha,\beta) \subseteq \mathrm{OpenProd}_R(\alpha,\beta)$. $\mathrm{OpenProd}_R(\alpha,\beta) \subseteq \mathrm{ClosedProd}_R(\alpha,\beta)$. □

(30) If $\alpha \subseteq \beta$, then $\mathrm{ClosedProd}_R(\alpha, \alpha) \subseteq \mathrm{ClosedProd}_R(\beta, \beta)$. The theorem is a consequence of (17).

3. The Preorder on the α-Day

Let us consider α. The functor $\mathbf{No}_{\mathrm{Ord}}\alpha$ yielding a binary relation is defined by

(Def. 12) *it* preserves **No** comparison on $\mathrm{Day}_{it}\alpha \times \mathrm{Day}_{it}\alpha$ and $it \subseteq \mathrm{Day}_{it}\alpha \times \mathrm{Day}_{it}\alpha$.

Note that $\mathbf{No}_{\mathrm{Ord}}\alpha$ is almost **No** order. The functor $\mathrm{Day}\alpha$ yielding a non empty subset of $\mathrm{Games}(\alpha)$ is defined by the term

(Def. 13) $\mathrm{Day}_{\mathbf{No}_{\mathrm{Ord}}\alpha}\alpha$.

4. Surreal Number as a Special Type of Abstract Game

Let us consider o. We say that o is surreal if and only if

(Def. 14) there exists α such that $o \in \mathrm{Day}\alpha$.

Let us note that $\langle \emptyset, \emptyset \rangle$ is surreal and there exists a set which is surreal. Let α be an ordinal number. Note that every element of $\mathrm{Day}\alpha$ is surreal. A surreal number is a surreal set. In the sequel x, y, z, t, r, l denote surreal numbers and X, Y, Z denote sets.

The functor $\mathbf{0}_{\mathbf{No}}$ yielding a surreal number is defined by the term

(Def. 15) $\langle \emptyset, \emptyset \rangle$.

Note that every surreal number is pair and every set which is surreal is also non empty.

Let X be a set. We say that X is surreal-membered if and only if

(Def. 16) if $o \in X$, then o is surreal.

One can check that there exists a set which is surreal-membered. Let us consider x. Observe that $\{x\}$ is surreal-membered and L_x is surreal-membered as a set and R_x is surreal-membered as a set. Let X, Y be surreal-membered sets. One can check that $X \cup Y$ is surreal-membered and $X \setminus Y$ is surreal-membered and $X \cap Y$ is surreal-membered and there exists a set which is non empty and surreal-membered.

5. The Preorder of Surreal Numbers

Let us consider x and y. We say that $x \leqslant y$ if and only if

(Def. 17) there exists α such that $x \leqslant_{\mathbf{No}_{\mathrm{Ord}}\alpha} y$.

Now we state the propositions:

(31) Let us consider ordinal numbers α, β, X. Suppose $X \subseteq \alpha$ and $X \subseteq \beta$. Then $\mathbf{No}_{\mathrm{Ord}}\alpha \cap (\mathrm{BeforeGames}(X) \times \mathrm{BeforeGames}(X)) = \mathbf{No}_{\mathrm{Ord}}\beta \cap (\mathrm{BeforeGames}(X) \times \mathrm{BeforeGames}(X))$. The theorem is a consequence of (17), (23), (29), and (20).

(32) Suppose $\alpha \subseteq \beta$. Then $\mathrm{ClosedProd}_{\mathbf{No}_{\mathrm{Ord}}\alpha}(\alpha, \alpha) = \mathrm{ClosedProd}_{\mathbf{No}_{\mathrm{Ord}}\beta}(\alpha, \alpha)$. The theorem is a consequence of (31) and (15).

(33) $\langle a, b \rangle \in \mathrm{ClosedProd}_{\mathbf{No}_{\mathrm{Ord}}\alpha}(\alpha, \alpha)$ if and only if $a, b \in \mathrm{Day}\alpha$.

(34) Suppose $\alpha \subseteq \beta$. Then $\mathbf{No}_{\mathrm{Ord}}\alpha = \mathbf{No}_{\mathrm{Ord}}\beta \cap \mathrm{ClosedProd}_{\mathbf{No}_{\mathrm{Ord}}\beta}(\alpha, \alpha)$. The theorem is a consequence of (30) and (23).

(35) If $\alpha \subseteq \beta$, then $\mathrm{Day}\alpha \subseteq \mathrm{Day}\beta$. The theorem is a consequence of (31), (10), and (9).

(36) If $o \in \mathrm{Day}_{\mathbf{No}_{\mathrm{Ord}}\alpha}\beta$ and $\beta \subseteq \alpha$, then $o \in \mathrm{Day}\beta$. The theorem is a consequence of (31) and (10).

Let us consider x. The functor $\mathfrak{born}\, x$ yielding an ordinal number is defined by

(Def. 18) $x \in \mathrm{Day}\, it$ and for every θ such that $x \in \mathrm{Day}\theta$ holds $it \subseteq \theta$.

Now we state the propositions:

(37) $\mathfrak{born}\, x = \emptyset$ if and only if $x = \mathbf{0}_{\mathbf{No}}$. The theorem is a consequence of (2) and (8).

(38) If $x \in \mathrm{Day}\alpha$, then $\mathfrak{born}\, x = \mathfrak{born}_{\mathbf{No}_{\mathrm{Ord}}\alpha} x$. The theorem is a consequence of (36), (31), and (11).

(39) If $a \leqslant_{\mathbf{No}_{\mathrm{Ord}}\alpha} b$ and $a, b \in \mathrm{Day}\beta$, then $a \leqslant_{\mathbf{No}_{\mathrm{Ord}}\beta} b$. The theorem is a consequence of (33), (32), (34), (30), and (23).

(40) $x \leqslant y$ if and only if for every α such that $x, y \in \mathrm{Day}\alpha$ holds $x \leqslant_{\mathbf{No}_{\mathrm{Ord}}\alpha} y$. The theorem is a consequence of (39) and (35).

Let L, R be sets. We say that $L \succeq R$ if and only if

(Def. 19) for every l and r such that $l \in L$ and $r \in R$ holds $r \leqslant l$.

Let R, L be sets. We introduce the notation $L \preceq R$ as a synonym of $R \succeq L$.

Let L, R be sets. We say that $L \ll R$ if and only if

(Def. 20) for every l and r such that $l \in L$ and $r \in R$ holds $r \not\leqslant l$.

We introduce the notation $R \gg L$ as a synonym of $L \ll R$. Now we state the propositions:

(41) Let us consider sets X_1, X_2, Y. If $X_1 \ll Y$ and $X_2 \ll Y$, then $X_1 \cup X_2 \ll Y$.

(42) Let us consider sets X, Y_1, Y_2. If $X \ll Y_1$ and $X \ll Y_2$, then $X \ll Y_1 \cup Y_2$.

(43) $x \leqslant y$ if and only if $\mathrm{L}_x \ll \{y\}$ and $\{x\} \ll \mathrm{R}_y$.

PROOF: Consider A_3 being an ordinal number such that $x \in \mathrm{Day} A_3$. Consider A_4 being an ordinal number such that $y \in \mathrm{Day} A_4$. Set $\alpha = A_3 \cup A_4$. $\mathrm{Day} A_3 \subseteq \mathrm{Day}\alpha$ and $\mathrm{Day} A_4 \subseteq \mathrm{Day}\alpha$. Set $S = \mathbf{No}_{\mathrm{Ord}}\alpha$. If $x \leqslant y$, then $\mathrm{L}_x \ll \{y\}$ and $\{x\} \ll \mathrm{R}_y$. $\langle x, y \rangle \in \mathrm{ClosedProd}_S(\alpha, \alpha)$. $\mathrm{L}_x \ll_S \{y\}$. $\{x\} \ll_S \mathrm{R}_y$. $\square$

(44) Let us consider sets X_1, X_2, Y_1, Y_2. Suppose for every x such that $x \in X_1$ there exists y such that $y \in X_2$ and $x \leqslant y$ and for every x such that $x \in Y_2$ there exists y such that $y \in Y_1$ and $y \leqslant x$ and $x = \langle X_1, Y_1 \rangle$ and $y = \langle X_2, Y_2 \rangle$. Then $x \leqslant y$. The theorem is a consequence of (43).

(45) $\mathrm{L}_x \ll \mathrm{R}_x$. The theorem is a consequence of (7), (35), (36), and (40).

(46) Let us consider sets X, Y, and α. Then $\langle X, Y \rangle \in \mathrm{Day}\alpha$ if and only if $X \ll Y$ and for every object o such that $o \in X \cup Y$ there exists θ such that $\theta \in \alpha$ and $o \in \mathrm{Day}\theta$. The theorem is a consequence of (45), (7), (36), (4), (33), (31), and (10).

(47) Suppose X is surreal-membered. Then there exists an ordinal number M such that for every o such that $o \in X$ there exists an ordinal number α such that $\alpha \in M$ and $o \in \mathrm{Day}\alpha$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_1$ is a surreal number and for every surreal number z such that $z = \$_1$ holds $\$_2 = \mathfrak{b}\mathrm{orn}\, z$. For every objects x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[x, z]$ holds $y = z$. Consider O_2 being a set such that for every object z, $z \in O_2$ iff there exists an object y such that $y \in X$ and $\mathcal{P}[y, z]$. For every set x such that $x \in O_2$ holds x is ordinal. $\square$

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