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Elementary Number Theory Problems. Part VII

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Summary. In this paper problems 48, 80, 87, 89, and 124 from [7] are formalized, using the Mizar formalism [1], [2], [4]. The work is natural continuation of [5] and [3] as suggested in [6].

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1. Preliminaries

From now on X denotes a set, a, b, c, k, m, n denote natural numbers, i, j denote integers, r denotes a real number, and p, p_1, p_2 denote prime numbers.

Now we state the propositions:

(1) $\gcd(m, m \cdot n) = m$.

(2) If $m \neq 1$, then m and $m \cdot n$ are not relatively prime.

(3) If $i \neq -1$ and $i \neq 1$ and $i \mid j$, then $i \nmid j+1$.

(4) If $i \neq -1$ and $i \neq 1$ and $i \mid j$, then $i \nmid j-1$.

(5) If $i \mid j$, then i and $j+1$ are relatively prime.
PROOF: For every integer m such that $m \mid i$ and $m \mid j+1$ holds $m \mid 1$ by [8, (1)]. □

(6) If $i \mid j$, then i and $j-1$ are relatively prime.
PROOF: For every integer m such that $m \mid i$ and $m \mid j-1$ holds $m \mid 1$. □

(7) If $a+b+c$ is odd and a, b, c are mutually coprime, then a is odd and b is odd and c is odd.

(8) (i) $4 \cdot n \bmod 8 = 0$, or

(ii) $4 \cdot n \bmod 8 = 4$.

(9) If $n \mid 2$, then $n = 1$ or $n = 2$.

(10) If $n \mid 6$, then $n = 1$ or $n = 2$ or $n = 3$ or $n = 6$.

(11) If $n \mid 9$, then $n = 1$ or $n = 3$ or $n = 9$.

(12) If $n \mid 10$, then $n = 1$ or $n = 2$ or $n = 5$ or $n = 10$.

(13) If $n \mid 25$, then $n = 1$ or $n = 5$ or $n = 25$.

(14) If $n \mid 26$, then $n = 1$ or $n = 2$ or $n = 13$ or $n = 26$.

(15) If $n \mid 36$, then $n = 1$ or $n = 2$ or $n = 3$ or $n = 4$ or $n = 6$ or $n = 9$ or $n = 12$ or $n = 18$ or $n = 36$.

(16) If $n \mid 50$, then $n = 1$ or $n = 2$ or $n = 5$ or $n = 10$ or $n = 25$ or $n = 50$.

(17) If $n \mid 65$, then $n = 1$ or $n = 5$ or $n = 13$ or $n = 65$.

(18) If $n \mid 82$, then $n = 1$ or $n = 2$ or $n = 41$ or $n = 82$.

(19) If $n \mid 122$, then $n = 1$ or $n = 2$ or $n = 61$ or $n = 122$.

(20) If $n \mid 145$, then $n = 1$ or $n = 5$ or $n = 29$ or $n = 145$.

(21) If $n \mid 226$, then $n = 1$ or $n = 2$ or $n = 113$ or $n = 226$.

(22) If $n \mid 325$, then $n = 1$ or $n = 5$ or $n = 13$ or $n = 25$ or $n = 65$ or $n = 325$.

(23) If $n \mid 362$, then $n = 1$ or $n = 2$ or $n = 181$ or $n = 362$.

(24) If $n \mid 485$, then $n = 1$ or $n = 5$ or $n = 97$ or $n = 485$.

(25) If $n \mid 626$, then $n = 1$ or $n = 2$ or $n = 313$ or $n = 626$.

(26) If $m \cdot n = p$, then $m = 1$ and $n = p$ or $m = p$ and $n = 1$.

(27) If $m \cdot n = 10$, then $m = 1$ and $n = 10$ or $m = 2$ and $n = 5$ or $m = 5$ and $n = 2$ or $m = 10$ and $n = 1$. The theorem is a consequence of (12).

(28) If $m \cdot n = 25$, then $m = 1$ and $n = 25$ or $m = 5$ and $n = 5$ or $m = 25$ and $n = 1$. The theorem is a consequence of (13).

(29) If $m \cdot n = 26$, then $m = 1$ and $n = 26$ or $m = 2$ and $n = 13$ or $m = 13$ and $n = 2$ or $m = 26$ and $n = 1$. The theorem is a consequence of (14).

(30) If $m \cdot n = 50$, then $m = 1$ and $n = 50$ or $m = 2$ and $n = 25$ or $m = 5$ and $n = 10$ or $m = 10$ and $n = 5$ or $m = 25$ and $n = 2$ or $m = 50$ and $n = 1$. The theorem is a consequence of (16).

(31) If $m \cdot n = 65$, then $m = 1$ and $n = 65$ or $m = 5$ and $n = 13$ or $m = 13$ and $n = 5$ or $m = 65$ and $n = 1$. The theorem is a consequence of (17).

(32) If $m \cdot n = 82$, then $m = 1$ and $n = 82$ or $m = 2$ and $n = 41$ or $m = 41$ and $n = 2$ or $m = 82$ and $n = 1$. The theorem is a consequence of (18).

(33) If $m \cdot n = 122$, then $m = 1$ and $n = 122$ or $m = 2$ and $n = 61$ or $m = 61$ and $n = 2$ or $m = 122$ and $n = 1$. The theorem is a consequence of (19).

(34) If $m \cdot n = 145$, then $m = 1$ and $n = 145$ or $m = 5$ and $n = 29$ or $m = 29$ and $n = 5$ or $m = 145$ and $n = 1$. The theorem is a consequence of (20).

(35) If $m \cdot n = 226$, then $m = 1$ and $n = 226$ or $m = 2$ and $n = 113$ or $m = 113$ and $n = 2$ or $m = 226$ and $n = 1$. The theorem is a consequence of (21).

(36) If $m \cdot n = 325$, then $m = 1$ and $n = 325$ or $m = 5$ and $n = 65$ or $m = 13$ and $n = 25$ or $m = 25$ and $n = 13$ or $m = 65$ and $n = 5$ or $m = 325$ and $n = 1$. The theorem is a consequence of (22).

(37) If $m \cdot n = 362$, then $m = 1$ and $n = 362$ or $m = 2$ and $n = 181$ or $m = 181$ and $n = 2$ or $m = 362$ and $n = 1$. The theorem is a consequence of (23).

(38) If $m \cdot n = 485$, then $m = 1$ and $n = 485$ or $m = 5$ and $n = 97$ or $m = 97$ and $n = 5$ or $m = 485$ and $n = 1$. The theorem is a consequence of (24).

(39) If $m \cdot n = 626$, then $m = 1$ and $n = 626$ or $m = 2$ and $n = 313$ or $m = 313$ and $n = 2$ or $m = 626$ and $n = 1$. The theorem is a consequence of (25).

(40) If $p_1 \neq p_2$, then $2 \leqslant p_1$ and $3 \leqslant p_2$ or $3 \leqslant p_1$ and $2 \leqslant p_2$.

2. PROBLEM 48

Let n be a natural number. We say that n satisfies Sierpiński Problem 48 if and only if

(Def. 1) there exist natural numbers a, b, c such that $n = a + b + c$ and $a > 1$ and $b > 1$ and $c > 1$ and a, b, c are mutually coprime.

Now we state the propositions:

(41) If n is even and $n > 8$, then n satisfies Sierpiński Problem 48. The theorem is a consequence of (5) and (6).

(42) If $n > 17$, then n satisfies Sierpiński Problem 48. The theorem is a consequence of (41), (10), (4), (11), (9), (6), (5), and (3).

(43) 17 doesn't satisfy Sierpiński Problem 48. The theorem is a consequence of (7) and (1).

3. Problem 80

Now we state the propositions:

(44) Let us consider prime numbers p, q, and a natural number n. Suppose $p \cdot (p+1) + q \cdot (q+1) = n \cdot (n+1)$. Then

(i) $p = 2$ and $q = 2$ and $n = 3$, or

(ii) $p = 5$ and $q = 3$ and $n = 6$, or

(iii) $p = 3$ and $q = 5$ and $n = 6$.

The theorem is a consequence of (26).

(45) Let us consider prime numbers p, q, r. If $p \cdot (p+1) + q \cdot (q+1) = r \cdot (r+1)$, then $p = q = 2$ and $r = 3$. The theorem is a consequence of (44).

4. Problem 87

Let n be a natural number. We say that n satisfies Sierpiński Problem 87a if and only if

(Def. 2) there exist prime numbers a, b, c such that a, b, c are mutually different and $n^2 + 1 = a \cdot b \cdot c$.

We say that n satisfies Sierpiński Problem 87b if and only if

(Def. 3) there exist odd prime numbers a, b, c such that a, b, c are mutually different and $n^2 + 1 = a \cdot b \cdot c$.

Now we state the propositions:

(46) $13^2 + 1 = 2 \cdot 5 \cdot 17$.

(47) 13 satisfies Sierpiński Problem 87a. The theorem is a consequence of (46).

(48) $17^2 + 1 = 2 \cdot 5 \cdot 29$.

(49) 17 satisfies Sierpiński Problem 87a. The theorem is a consequence of (48).

(50) $21^2 + 1 = 2 \cdot 13 \cdot 17$.

(51) 21 satisfies Sierpiński Problem 87a. The theorem is a consequence of (50).

(52) $23^2 + 1 = 2 \cdot 5 \cdot 53$.

(53) 23 satisfies Sierpiński Problem 87a. The theorem is a consequence of (52).

(54) $27^2 + 1 = 2 \cdot 5 \cdot 73$.

(55) 27 satisfies Sierpiński Problem 87a. The theorem is a consequence of (54).

(56) If n satisfies Sierpiński Problem 87a and $n \leqslant 27$, then $n \in \{13, 17, 21, 23, 27\}$.

(57) $112^2 + 1 = 5 \cdot 13 \cdot 193$.

(58) 112 satisfies Sierpiński Problem 87b. The theorem is a consequence of (57).

5. Problem 89

Let us consider n. We say that n has exactly two different prime divisors if and only if

(Def. 4) there exist prime numbers p, q such that $p \neq q$ and $p \mid n$ and $q \mid n$ and for every prime number r such that $r \neq p$ and $r \neq q$ holds $r \nmid n$.

Let n be a complex number. We say that n is a product of two different primes if and only if

(Def. 5) there exist prime numbers p, q such that $p \neq q$ and $n = p \cdot q$.

Now we state the propositions:

(59) Let us consider prime numbers p, q, and natural numbers a, b. Suppose $a \neq 1$ and $b \neq 1$ and $p \cdot q = a \cdot b$. Then

(i) $p = a$ and $q = b$, or

(ii) $p = b$ and $q = a$.

(60) If n is a product of two different primes, then for every a and b such that $a \neq 1$ and $b \neq 1$ and $n = a \cdot b$ holds a is prime and b is prime.

(61) p is not a product of two different primes.

(62) If $p_1 \neq p_2$, then $p_1 \cdot p_2$ is a product of two different primes.

(63) If $a \neq 1$ and $a \neq n$ and a is not prime and $a \mid n$, then n is not a product of two different primes.

(64) $p \cdot p$ is not a product of two different primes.

(65) If n is a product of two different primes, then $n \geqslant 6$. The theorem is a consequence of (40).

Let us consider n. We say that n satisfies Sierpiński Problem 89 if and only if

(Def. 6) n is a product of two different primes and $n + 1$ is a product of two different primes and $n + 2$ is a product of two different primes.

Now we state the propositions:

(66) 33 satisfies Sierpiński Problem 89.

(67) 85 satisfies Sierpiński Problem 89.

(68) 93 satisfies Sierpiński Problem 89.

(69) 141 satisfies Sierpiński Problem 89.

(70) 201 satisfies Sierpiński Problem 89.

(71) If n satisfies Sierpiński Problem 89 and $n \leqslant 201$, then $n \in \{33, 85, 93, 141, 201\}$.

(72) There exists no n such that n satisfies Sierpiński Problem 89 and $n+1$ satisfies Sierpiński Problem 89 and $n+2$ satisfies Sierpiński Problem 89 and $n+3$ satisfies Sierpiński Problem 89.

(73) (i) $33 = 3 \cdot 11$, and

(ii) 33 has exactly two different prime divisors.

(74) (i) $34 = 2 \cdot 17$, and

(ii) 34 has exactly two different prime divisors.

(75) (i) $35 = 5 \cdot 7$, and

(ii) 35 has exactly two different prime divisors.

(76) (i) $36 = 2 \cdot 2 \cdot 3 \cdot 3$, and

(ii) 36 has exactly two different prime divisors.

The theorem is a consequence of (15).

6. Problem 124

Now we state the propositions:

(77) If $n = 28 \cdot k + 1$, then $29 \mid (2^{2 \cdot n} + 1)^2 + 2^2$.

(78) If $k > 0$ and $n = 28 \cdot k + 1$, then $(2^{2 \cdot n} + 1)^2 + 2^2$ is composite. The theorem is a consequence of (77).

(79) $\{(2^{2 \cdot n} + 1)^2 + 2^2$, where n is a natural number : $(2^{2 \cdot n} + 1)^2 + 2^2$ is composite$\}$ is infinite.

Proof: Set $X = \{(2^{2 \cdot n} + 1)^2 + 2^2$, where n is a natural number : $(2^{2 \cdot n} + 1)^2 + 2^2$ is composite$\}$. Set $n = 28 \cdot 1 + 1$. $(2^{2 \cdot n} + 1)^2 + 2^2$ is composite. X is natural-membered. For every a such that $a \in X$ there exists b such that $b > a$ and $b \in X$. □

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