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Finite Dimensional Real Normed Spaces are Proper Metric Spaces[1]

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Karuizawa Hotch 244-1
Nagano, Japan

Summary. In this article, we formalize in Mizar [1], [2] the topological properties of finite-dimensional real normed spaces. In the first section, we formalize the Bolzano-Weierstrass theorem, which states that a bounded sequence of points in an n-dimensional Euclidean space has a certain subsequence that converges to a point. As a corollary, it is also shown the equivalence between a subset of an n-dimensional Euclidean space being compact and being closed and bounded.

In the next section, we formalize the definitions of L1-norm (Manhattan Norm) and maximum norm and show their topological equivalence in n-dimensional Euclidean spaces and finite-dimensional real linear spaces. In the last section, we formalize the linear isometries and their topological properties. Namely, it is shown that a linear isometry between real normed spaces preserves properties such as continuity, the convergence of a sequence, openness, closeness, and compactness of subsets. Finally, it is shown that finite-dimensional real normed spaces are proper metric spaces. We referred to [5], [9], and [7] in the formalization.

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1. Bolzano-Weierstrass Theorem and its Corollary

From now on X denotes a set, n, m, k denote natural numbers, K denotes a field, f denotes an n-element, real-valued finite sequence, and M denotes a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Now we state the propositions:

(1) Let us consider an element x of $\mathcal{R}^{n+1}$, and an element y of $\mathcal{R}^n$. If $y = x {\restriction} n$, then $|y| \leqslant |x|$.

(2) Let us consider an element x of $\mathcal{R}^{n+1}$, and an element w of $\mathbb{R}$. If $w = x(n+1)$, then $|w| \leqslant |x|$.

(3) Let us consider an element x of $\mathcal{R}^{n+1}$, an element y of $\mathcal{R}^n$, and an element w of $\mathbb{R}$. If $y = x {\restriction} n$ and $w = x(n+1)$, then $|x| \leqslant |y| + |w|$.

(4) Let us consider elements x, y of $\mathcal{R}^n$, and a natural number m. If $m \leqslant n$, then $(x - y) {\restriction} m = x {\restriction} m - y {\restriction} m$.

(5) Let us consider a natural number n, and a sequence x of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose there exists a real number K such that for every natural number i, $\|x(i)\| < K$. Then there exists a subsequence x_0 of x such that x_0 is convergent.
Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every sequence x of $\langle \mathcal{E}^{\$_1}, \| \cdot \| \rangle$ such that there exists a real number K such that for every natural number i, $\|x(i)\| < K$ there exists a subsequence x_0 of x such that x_0 is convergent. $\mathcal{P}[0]$ by [4, (18)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(6) Let us consider a real normed space N, and a subset X of N. Suppose X is compact. Then
   (i) X is closed, and
   (ii) there exists a real number r such that for every point y of N such that $y \in X$ holds $\|y\| < r$.

(7) Let us consider a subset X of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then X is compact if and only if X is closed and there exists a real number r such that for every point y of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $y \in X$ holds $\|y\| < r$.

2. L1-norm and Maximum Norm

Now we state the propositions:

(8) Let us consider a non empty natural number n, and an element x of $\mathcal{R}^n$. Then there exists a real number x_4 such that
   (i) $x_4 \in \operatorname{rng} |x|$, and
   (ii) for every natural number i such that $i \in \operatorname{dom} x$ holds $|x|(i) \leqslant x_4$.

PROOF: Set $F = \text{rng}|x|$. Set $x_4 = \sup F$. For every natural number i such that $i \in \text{dom}\, x$ holds $|x|(i) \leqslant x_4$. □

(9) Let us consider a real-valued finite sequence x. Then $0 \leqslant \sum|x|$.

Let n be a natural number. Assume n is not empty. The functor max-norm(n) yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 1) for every element x of $\mathcal{R}^n$, $it(x) \in \text{rng}|x|$ and for every natural number i such that $i \in \text{dom}\, x$ holds $|x|(i) \leqslant it(x)$.

Assume n is not empty. The functor sum-norm(n) yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by

(Def. 2) for every element x of $\mathcal{R}^n$, $it(x) = \sum|x|$.

Now we state the proposition:

(10) Let us consider an element x of $\mathcal{R}^n$, and a real number x_4. Suppose $x_4 \in \text{rng}|x|$ and for every natural number i such that $i \in \text{dom}\, x$ holds $|x|(i) \leqslant x_4$. Then

(i) $\sum|x| \leqslant n \cdot x_4$, and

(ii) $x_4 \leqslant |x| \leqslant \sum|x|$.

PROOF: Set $F = n \mapsto x_4$. For every natural number j such that $j \in \text{Seg}\, n$ holds $|x|(j) \leqslant F(j)$. Consider i being an object such that $i \in \text{dom}|x|$ and $x_4 = |x|(i)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element x of $\mathcal{R}^{\$_1}$, $|x|^{\mathbf{2}} \leqslant (\sum|x|)^{\mathbf{2}}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

Let us consider a non empty natural number n, elements x, y of $\mathcal{R}^n$, and a real number a. Now we state the propositions:

(11) (i) $0 \leqslant (\text{max-norm}(n))(x)$, and

(ii) $(\text{max-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, and

(iii) $(\text{max-norm}(n))(a \cdot x) = |a| \cdot (\text{max-norm}(n))(x)$, and

(iv) $(\text{max-norm}(n))(x + y) \leqslant (\text{max-norm}(n))(x) + (\text{max-norm}(n))(y)$.

PROOF: Set $x_4 = (\text{max-norm}(n))(x)$. Set $y_2 = (\text{max-norm}(n))(y)$. Consider j_0 being an object such that $j_0 \in \text{dom}|x|$ and $x_4 = |x|(j_0)$. Consider k_0 being an object such that $k_0 \in \text{dom}|y|$ and $y_2 = |y|(k_0)$. $(\text{max-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. $(\text{max-norm}(n))(a \cdot x) = |a| \cdot (\text{max-norm}(n))(x)$. $(\text{max-norm}(n))(x + y) \leqslant (\text{max-norm}(n))(x) + (\text{max-norm}(n))(y)$. □

(12) (i) $0 \leqslant (\text{sum-norm}(n))(x)$, and

(ii) $(\text{sum-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, and

(iii) $(\text{sum-norm}(n))(a \cdot x) = |a| \cdot (\text{sum-norm}(n))(x)$, and

(iv) $(\text{sum-norm}(n))(x + y) \leqslant (\text{sum-norm}(n))(x) + (\text{sum-norm}(n))(y)$.

PROOF: $0 \leqslant \sum|x|$. $(\text{sum-norm}(n))(x) = 0$ iff $x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. For every natural number j such that $j \in \operatorname{Seg} n$ holds $|x + y|(j) \leqslant (|x| + |y|)(j)$. □

(13) Let us consider a non empty natural number n, and an element x of $\mathcal{R}^n$. Then

(i) $(\text{sum-norm}(n))(x) \leqslant n \cdot (\text{max-norm}(n))(x)$, and

(ii) $(\text{max-norm}(n))(x) \leqslant |x| \leqslant (\text{sum-norm}(n))(x)$.

The theorem is a consequence of (10).

(14) The RLS structure of $\langle \mathcal{E}^n, \| \cdot \| \rangle = \mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$.

(15) Let us consider a real number a, elements x, y of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and elements x_0, y_0 of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$. Suppose $x = x_0$ and $y = y_0$. Then

(i) the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle =$ the carrier of $\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}$, and

(ii) $0_{\langle \mathcal{E}^n, \| \cdot \| \rangle} = 0_{\mathbb{R}_{\mathbb{R}}^{\operatorname{Seg} n}}$, and

(iii) $x + y = x_0 + y_0$, and

(iv) $a \cdot x = a \cdot x_0$, and

(v) $-x = -x_0$, and

(vi) $x - y = x_0 - y_0$.

The theorem is a consequence of (14).

Let X be a finite dimensional real linear space.

One can check that RLSp2RVSp(X) is finite dimensional.

Now we state the proposition:

(16) Let us consider a finite dimensional real linear space X, an ordered basis b of RLSp2RVSp(X), and an element y of RLSp2RVSp(X). Then $y \rightarrow b$ is an element of $\mathcal{R}^{\dim(X)}$.

Let X be a finite dimensional real linear space and b be an ordered basis of RLSp2RVSp(X). The functor max-norm(X, b) yielding a function from X into $\mathbb{R}$ is defined by

(Def. 3) for every element x of X, there exists an element y of RLSp2RVSp(X) and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \rightarrow b$ and $it(x) = (\text{max-norm}(\dim(X)))(z)$.

The functor sum-norm(X, b) yielding a function from X into $\mathbb{R}$ is defined by

(Def. 4) for every element x of X, there exists an element y of RLSp2RVSp(X) and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \rightarrow b$ and $it(x) = (\text{sum-norm}(\dim(X)))(z)$.

The functor Euclid-norm(X, b) yielding a function from X into $\mathbb{R}$ is defined by

(Def. 5) for every element x of X, there exists an element y of RLSp2RVSp(X) and there exists an element z of $\mathcal{R}^{\dim(X)}$ such that $x = y$ and $z = y \to b$ and $it(x) = |z|$.

Now we state the proposition:

(17) Let us consider a natural number n, an element a of $\mathbb{R}$, an element a_1 of $\mathbb{R}_{\rm F}$, elements x, y of $\mathcal{R}^n$, and elements x_1, y_1 of (the carrier of $\mathbb{R}_{\rm F})^n$. Suppose $a = a_1$ and $x = x_1$ and $y = y_1$. Then

(i) $a \cdot x = a_1 \cdot x_1$, and

(ii) $x + y = x_1 + y_1$.

Let us consider a finite dimensional real linear space X, an ordered basis b of RLSp2RVSp(X), elements x, y of X, and a real number a. Now we state the propositions:

(18) Suppose $\dim(X) \neq 0$. Then

(i) $0 \leqslant (\text{max-norm}(X, b))(x)$, and

(ii) $(\text{max-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{max-norm}(X, b))(a \cdot x) = |a| \cdot (\text{max-norm}(X, b))(x)$, and

(iv) $(\text{max-norm}(X, b))(x+y) \leqslant (\text{max-norm}(X, b))(x) + (\text{max-norm}(X, b))(y)$.

The theorem is a consequence of (11).

(19) Suppose $\dim(X) \neq 0$. Then

(i) $0 \leqslant (\text{sum-norm}(X, b))(x)$, and

(ii) $(\text{sum-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{sum-norm}(X, b))(a \cdot x) = |a| \cdot (\text{sum-norm}(X, b))(x)$, and

(iv) $(\text{sum-norm}(X, b))(x + y) \leqslant (\text{sum-norm}(X, b))(x) + (\text{sum-norm}(X, b))(y)$.

The theorem is a consequence of (12).

(20) (i) $0 \leqslant (\text{Euclid-norm}(X, b))(x)$, and

(ii) $(\text{Euclid-norm}(X, b))(x) = 0$ iff $x = 0_X$, and

(iii) $(\text{Euclid-norm}(X, b))(a \cdot x) = |a| \cdot (\text{Euclid-norm}(X, b))(x)$, and

(iv) $(\text{Euclid-norm}(X, b))(x+y) \leqslant (\text{Euclid-norm}(X, b))(x) + (\text{Euclid-norm}(X, b))(y)$.

(21) Let us consider a finite dimensional real linear space X, an ordered basis b of RLSp2RVSp(X), and an element x of X. Suppose $\dim(X) \neq 0$. Then

(i) $(\text{sum-norm}(X, b))(x) \leqslant (\dim(X)) \cdot (\text{max-norm}(X, b))(x)$, and

(ii) $(\text{max-norm}(X, b))(x) \leqslant (\text{Euclid-norm}(X, b))(x) \leqslant (\text{sum-norm}(X, b))(x)$.

The theorem is a consequence of (13).

(22) Let us consider a finite dimensional real linear space V, and an ordered basis b of RLSp2RVSp(V). Suppose $\dim(V) \neq 0$. Then there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that

(i) S is bijective, and

(ii) for every element x of RLSp2RVSp(V), $S(x) = x \rightarrow b$.

The theorem is a consequence of (15).

(23) Let us consider a finite dimensional real normed space V. Suppose $\dim(V) \neq 0$. Then there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ and there exists a finite dimensional vector space W over $\mathbb{R}_{\mathrm{F}}$ and there exists an ordered basis b of W such that $W =$ RLSp2RVSp(V) and S is bijective and for every element x of W, $S(x) = x \rightarrow b$. The theorem is a consequence of (15).

(24) Let us consider a real normed space V, a finite dimensional real linear space W, and an ordered basis b of RLSp2RVSp(W). Suppose V is finite dimensional and $\dim(V) \neq 0$ and the RLS structure of $V =$ the RLS structure of W. Then there exist real numbers k_1, k_2 such that

(i) $0 < k_1$, and

(ii) $0 < k_2$, and

(iii) for every point x of V, $\|x\| \leqslant k_1 \cdot (\text{max-norm}(W, b))(x)$ and $(\text{max-norm}(W, b))(x) \leqslant k_2 \cdot \|x\|$.

Proof: Reconsider $e = b$ as a finite sequence of elements of W. Reconsider $e_1 = e$ as a finite sequence of elements of V. Define $\mathcal{F}(\text{natural number}) = \|e_{1/\$_1}\| (\in \mathbb{R})$. Consider k being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{len} k = \operatorname{len} b$ and for every natural number i such that $i \in \operatorname{dom} k$ holds $k(i) = \mathcal{F}(i)$. Set $k_1 = \sum k$. For every natural number i such that $i \in \operatorname{dom} k$ holds $0 \leqslant k(i)$. For every point x of V, $\|x\| \leqslant (k_1 + 1) \cdot (\text{max-norm}(W, b))(x)$ by [6, (12), (15)], [8, (7)].

Consider S_0 being a linear operator from W into $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that S_0 is bijective and for every element x of RLSp2RVSp(W), $S_0(x) = x \rightarrow b$. Reconsider $S = S_0$ as a function from the carrier of V into the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. For every elements x, y of V, $S(x+y) = S(x)+S(y)$. For every real number a and for every vector x of V, $S(a \cdot x) = a \cdot S(x)$.

Consider T being a linear operator from $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ into V such that $T = S^{-1}$ and T is one-to-one and onto. For every element x of V, $\|x\| \leqslant (k_1+1) \cdot \|S(x)\|$. For every element y of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$, $\|T(y)\| \leqslant (k_1+1) \cdot \|y\|$. Set $C_2 = \{y$, where y is an element of V : (max-norm$(W, b))(y) = 1\}$.

Set $C_1 = \{x$, where x is an element of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$: (max-norm(dim $(W)))(x) = 1\}$. For every object z such that $z \in C_2$ holds $z \in$ the carrier of V. For every object z such that $z \in C_1$ holds $z \in$ the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. Consider z_5 being a point of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $z_5 \neq 0_{\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle}$. Reconsider $z_6 = z_5$ as an element of $\mathcal{R}^{\dim(W)}$. (max-norm(dim $(W)))(z_6) \neq 0$. $0 <$ (max-norm(dim$(W)))(z_5)$. For every object y, $y \in T^{\circ} C_1$ iff $y \in C_2$. Reconsider $g =$ max-norm(dim$(W))$ as a function from the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ into $\mathbb{R}$. Set $D =$ the carrier of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$. For every point x_0 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ and for every real number r such that $x_0 \in D$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $x_1 \in D$ and $\|x_1 - x_0\| < s$ holds $|g_{/x_1} - g_{/x_0}| < r$.

For every sequence s_1 of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $\operatorname{rng} s_1 \subseteq C_1$ and s_1 is convergent holds $\lim s_1 \in C_1$. There exists a real number r such that for every point y of $\langle \mathcal{E}^{\dim(W)}, \|\cdot\| \rangle$ such that $y \in C_1$ holds $\|y\| < r$ by (13), [3, (1)]. Reconsider $f = \mathrm{id}_{C_2}$ as a partial function from V to V. Consider y_0 being an element of V such that $y_0 \in \operatorname{dom}\|f\|$ and $\inf \operatorname{rng}\|f\| = \|f\|(y_0)$. Set $k_2 = \|f_{/y_0}\|$. For every element x of V such that $x \in C_2$ holds $k_2 \leqslant \|x\|$. $k_2 \neq 0$. For every point x of V, (max-norm$(W, b))(x) \leqslant \frac{1}{k_2} \cdot \|x\|$. □

(25) Let us consider real normed spaces X, Y. Suppose the RLS structure of $X =$ the RLS structure of Y and X is finite dimensional and $\dim(X) \neq 0$. Then there exist real numbers k_1, k_2 such that

(i) $0 < k_1$, and

(ii) $0 < k_2$, and

(iii) for every element x of X and for every element y of Y such that $x = y$ holds $\|x\| \leqslant k_1 \cdot \|y\|$ and $\|y\| \leqslant k_2 \cdot \|x\|$.

The theorem is a consequence of (24).

(26) Let us consider a real normed space V. Suppose V is finite dimensional and $\dim(V) \neq 0$. Then there exist real numbers k_1, k_2 and there exists a linear operator S from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that S is bijective and $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|S(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|S(x)\|$. The theorem is a consequence of (23), (24), and (21).

3. Linear Isometry and its Topological Properties

Let V, W be real normed spaces and L be a linear operator from V into W. We say that L is isometric-like if and only if

(Def. 6) there exist real numbers k_1, k_2 such that $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|L(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|L(x)\|$.

Now we state the proposition:

(27) Let us consider a finite dimensional real normed space V. Suppose $\dim(V) \neq 0$. Then there exists a linear operator L from V into $\langle \mathcal{E}^{\dim(V)}, \|\cdot\| \rangle$ such that L is one-to-one, onto, and isometric-like.

The theorem is a consequence of (26).

Let us consider real normed spaces V, W and a linear operator L from V into W. Now we state the propositions:

(28) Suppose L is one-to-one, onto, and isometric-like. Then there exists a linear operator K from W into V such that

(i) $K = L^{-1}$, and

(ii) K is one-to-one, onto, and isometric-like.

Proof: Consider K being a linear operator from W into V such that $K = L^{-1}$ and K is one-to-one and onto. Consider k_1, k_2 being real numbers such that $0 \leqslant k_1$ and $0 \leqslant k_2$ and for every element x of V, $\|L(x)\| \leqslant k_1 \cdot \|x\|$ and $\|x\| \leqslant k_2 \cdot \|L(x)\|$. For every element y of W, $\|K(y)\| \leqslant k_2 \cdot \|y\|$ and $\|y\| \leqslant k_1 \cdot \|K(y)\|$. □

(29) If L is one-to-one, onto, and isometric-like, then L is Lipschitzian.

(30) If L is one-to-one, onto, and isometric-like, then L is continuous on the carrier of V.

(31) Let us consider real normed spaces S, T, a linear operator I from S into T, and a point x of S. If I is one-to-one, onto, and isometric-like, then I is continuous in x.

The theorem is a consequence of (29).

(32) Let us consider real normed spaces S, T, a linear operator I from S into T, and a subset Z of S. If I is one-to-one, onto, and isometric-like, then I is continuous on Z.

The theorem is a consequence of (31).

Let us consider real normed spaces S, T, a linear operator I from S into T, and a sequence s_1 of S. Now we state the propositions:

(33) Suppose I is one-to-one, onto, and isometric-like and s_1 is convergent. Then

(i) $I \cdot s_1$ is convergent, and

(ii) $\lim I \cdot s_1 = I(\lim s_1)$.

The theorem is a consequence of (31).

(34) If I is one-to-one, onto, and isometric-like, then s_1 is convergent iff $I \cdot s_1$ is convergent. The theorem is a consequence of (28) and (33).

Let us consider real normed spaces S, T, a linear operator I from S into T, and a subset Z of S. Now we state the propositions:

(35) If I is one-to-one, onto, and isometric-like, then Z is closed iff $I^\circ Z$ is closed.

Proof: Consider J being a linear operator from T into S such that $J = I^{-1}$ and J is one-to-one, onto, and isometric-like. Z is closed iff $I^\circ Z$ is closed. □

(36) If I is one-to-one, onto, and isometric-like, then Z is open iff $I^\circ Z$ is open. The theorem is a consequence of (28) and (35).

(37) If I is one-to-one, onto, and isometric-like, then Z is compact iff $I^\circ Z$ is compact.

Proof: Consider J being a linear operator from T into S such that $J = I^{-1}$ and J is one-to-one, onto, and isometric-like. If $I^\circ Z$ is compact, then Z is compact. □

(38) Let us consider a finite dimensional real normed space V, and a subset X of V. Suppose $\dim(V) \neq 0$. Then X is compact if and only if X is closed and there exists a real number r such that for every point y of V such that $y \in X$ holds $\|y\| < r$. The theorem is a consequence of (6), (27), (35), and (37).

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