FORMALIZED MATHEMATICS
Vol. 28, No. 2, Pages 197–210, 2020
DOI: 10.2478/forma-2020-0017

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Multiplication-Related Classes of Complex Numbers

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Summary. The use of registrations is useful in shortening Mizar proofs [1], [2], both in terms of formalization time and article space. The proposed system of classes for complex numbers aims to facilitate proofs involving basic arithmetical operations and order checking. It seems likely that the use of self-explanatory adjectives could also improve legibility of these proofs, which would be an important achievement [3]. Additionally, some potentially useful definitions, following those defined for real numbers, are introduced.

MSC: 40-04 68V20

Keywords: complex numbers; multiplication; order

MML identifier: COMPLEX3, version: 8.1.10 5.63.1382

Let a be a complex number. One can check that $(a^{-1})^{-1}$ reduces to a.

We say that a is heavy if and only if

(Def. 1) $|a| > 1$.

We say that a is light if and only if

(Def. 2) $|a| < 1$.

We say that a is weightless if and only if

(Def. 3) $|a| = 0$ or $|a| = 1$.

Let us consider a real number a. Now we state the propositions:

(1) (i) a is heavy and negative iff $a < -1$, and

(ii) a is light and negative iff $-1 < a < 0$, and

(iii) a is light and positive iff $0 < a < 1$, and

(iv) a is heavy and positive iff $a > 1$, and

(v) a is weightless and positive iff $a = 1$, and

(vi) a is weightless and negative iff $a = -1$.

(2) (i) a is non light and negative iff $a \leqslant -1$, and

(ii) a is non heavy and negative iff $-1 \leqslant a < 0$, and

(iii) a is non heavy and positive iff $0 < a \leqslant 1$, and

(iv) a is non light and positive iff $1 \leqslant a$.

(3) a is weightless if and only if $a = \mathrm{sgn}(a)$.

PROOF: If a is weightless, then $a = \mathrm{sgn}(a)$. If $a = \mathrm{sgn}(a)$, then a is weightless. □

Let us note that every complex number which is zero is also weightless and every complex number which is heavy is also non light and every complex number which is non light is also non zero and every complex number which is heavy is also non weightless and every non zero complex number which is light is also non weightless and every integer which is light is also zero.

Every natural number which is trivial is also weightless and every natural number which is non heavy is also trivial and every natural number which is non zero is also non light and every natural number which is non trivial is also heavy and every complex number which is weightless is also non heavy and every complex number which is light is also non heavy and every non negative real number which is non light is also positive.

There exists a positive real number which is heavy and there exists a negative real number which is heavy and there exists a positive real number which is light and there exists a negative real number which is light and there exists a weightless integer which is positive and there exists a weightless integer which is negative.

Let us consider a complex number a. Now we state the propositions:

(4) $\Re(a) \geqslant -|a|$.

(5) $\Im(a) \geqslant -|a|$.

(6) $|\Re(a)| + |\Im(a)| \geqslant |a|$.

Let a be a complex number. Let us observe that $a \cdot (a^{-1})$ is trivial and $a \cdot \overline{a}$ is real and $a \cdot \overline{a}^2$ is non negative and $\frac{a}{|a|}$ is weightless.

The functor director(a) yielding a weightless complex number is defined by the term

(Def. 4) $\frac{a}{|a|}$.

Let us consider a complex number a. Now we state the propositions:

(7) $a = |a| \cdot \mathrm{director}(a)$.

(8) $\mathrm{director}(-a) = -\mathrm{director}(a)$.

Let a be a real number. We identify $\operatorname{sgn}(a)$ with $\operatorname{director}(a)$. Observe that $\operatorname{director}(a)$ is integer.

Let a be a negative real number. One can verify that $\operatorname{director}(a)$ is negative.

Let a be a positive real number. Note that $\operatorname{director}(a)$ is positive.

Let us note that $\operatorname{director}(0)$ reduces to 0.

Let a be a non weightless complex number. Let us note that $|a|$ is positive and $-a$ is non weightless and $\overline{a}$ is non weightless and a^{-1} is non weightless.

Let a be a weightless complex number. Observe that $-a$ is weightless and $\overline{a}$ is weightless and a^{-1} is weightless and $a \cdot \overline{a}$ is weightless and $|\Re(a)|$ is non heavy and $|\Im(a)|$ is non heavy and $|a|-1$ is weightless and $1-|a|$ is weightless.

Let a be a weightless real number. One can verify that $\operatorname{sgn}(a)$ reduces to a.

Let a be a heavy complex number. One can verify that $-a$ is heavy and $\overline{a}$ is heavy and a^{-1} is light and $a \cdot \overline{a}$ is heavy and $|\Re(a)|+|\Im(a)|$ is heavy and $|a|-1$ is positive and $1-|a|$ is negative.

Let a be a non light complex number. Note that $-a$ is non light and $\overline{a}$ is non light and a^{-1} is non heavy and $a \cdot \overline{a}$ is non light and $|\Re(a)|+|\Im(a)|$ is non light and $|a|-1$ is non negative and $1-|a|$ is non positive.

Let a be a light complex number. Observe that $-a$ is light and $\overline{a}$ is light and $a \cdot \overline{a}$ is light and $|a|-1$ is negative and $1-|a|$ is positive and $\Re(a)$ is light and $\Im(a)$ is light and $\Re(a)-1$ is negative and $\Re(a)-2$ is heavy and $\Im(a)-1$ is negative and $\Im(a)-2$ is heavy.

Let a be a non zero, light complex number. Note that a^{-1} is heavy.

Let a be a non heavy complex number. Let us note that $-a$ is non heavy and $\overline{a}$ is non heavy and $a \cdot \overline{a}$ is non heavy and $|a|-1$ is non positive and $1-|a|$ is non negative and $\Re(a)$ is non heavy and $\Im(a)$ is non heavy and $\Re(a)-1$ is non positive and $\Im(a)-1$ is non positive.

Let a be a non zero, non heavy complex number. Let us observe that a^{-1} is non light.

Let a be a complex number. The functor $\operatorname{rsgn}(a)$ yielding a non heavy complex number is defined by the term

(Def. 5) $\Re(\operatorname{director}(a))$.

The functor $\operatorname{isgn}(a)$ yielding a non heavy complex number is defined by the term

(Def. 6) $\Im(\operatorname{director}(a))$.

Let a be a real number. We identify $\operatorname{sgn}(a)$ with $\operatorname{rsgn}(a)$. One can check that $\operatorname{isgn}(a)$ is zero and $\operatorname{frac} a$ is light and $|a|+a$ is non negative and $|a|-a$ is non negative.

Let a be a heavy, positive real number. Observe that $a-1$ is positive and $1-a$ is negative.

Let a be a light, positive real number. One can check that $a-1$ is negative and $1-a$ is positive.

Now we state the propositions:

(9) Every non heavy complex number is light or weightless.

(10) Every non light complex number is heavy or weightless.

(11) Let us consider a heavy, positive real number a, and a non heavy real number b. Then $a > b > -a$. The theorem is a consequence of (1).

(12) Let us consider a non light, positive real number a, and a light real number b. Then $a > b > -a$. The theorem is a consequence of (1).

Let a be a heavy complex number and b be a non light complex number. Observe that $a \cdot b$ is heavy.

Let a, b be non light complex numbers. Note that $a \cdot b$ is non light.

Let a be a light complex number and b be a non heavy complex number. One can check that $a \cdot b$ is light.

Let a, b be non heavy complex numbers. Let us observe that $a \cdot b$ is non heavy.

Let a, b be weightless complex numbers. Let us note that $a \cdot b$ is weightless.

Let a be a complex number. The functor $\mathrm{cfrac}(a)$ yielding a light complex number is defined by the term

(Def. 7) $\mathrm{director}(a) \cdot \mathrm{frac}\,|a|$.

Now we state the proposition:

(13) Let us consider a complex number a. Then $\mathrm{cfrac}(-a) = -\mathrm{cfrac}(a)$. The theorem is a consequence of (8).

Let a be a non negative real number. We identify $\mathrm{cfrac}(a)$ with $\mathrm{frac}\,a$. Now we state the proposition:

(14) Let us consider a complex number a, and a natural number n. Then $|a|^n = |a^n|$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv |a|^{\$_1} = |a^{\$_1}|$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number l, $\mathcal{P}[l]$. □

Let a be a weightless complex number and n be a natural number. One can check that a^n is weightless.

Let a be a weightless real number. One can verify that $a^{2 \cdot n} - 1$ is weightless.

Let a be a non light complex number. Let us note that a^n is non light.

Let a be a non light real number. One can check that $a^{2 \cdot n} - 1$ is non negative.

Let a be a light complex number and n be a non zero natural number. Note that a^n is light and $\sqrt[n]{a}$ is light.

Let a be a light real number. Let us observe that $a^{2 \cdot n} - 1$ is negative.

Let a be a non heavy complex number and n be a natural number. One can check that a^n is non heavy.

Let a be a non heavy real number. Observe that $a^{2 \cdot n} - 1$ is non positive.

Let a be a heavy complex number and n be a non zero natural number. Let us observe that a^n is heavy and $\sqrt[n]{a}$ is heavy.

Let a be a non weightless complex number. One can check that a^n is non weightless.

Let a be a weightless complex number. Let us observe that $\sqrt[n]{a}$ is weightless.

Let a be a non weightless complex number. Observe that $\sqrt[n]{a}$ is non weightless.

Let a be a non light complex number. Note that $\sqrt[n]{a}$ is non light.

Let a be a non heavy complex number. One can verify that $\sqrt[n]{a}$ is non heavy.

Let a, b be weightless complex numbers. Observe that $\frac{a}{b}$ is weightless.

Let a be a non heavy complex number and b be a heavy complex number. Observe that $\frac{a}{b}$ is light.

Let a be a light complex number and b be a non light complex number. Observe that $\frac{a}{b}$ is light.

Let a be a non light complex number and b be a non zero, light complex number. Let us observe that $\frac{a}{b}$ is heavy.

Let a be a heavy complex number and b be a non zero, non heavy complex number. One can verify that $\frac{a}{b}$ is heavy.

Let a be a heavy, positive real number and b be a non negative real number. Note that $a + b$ is heavy.

Let a be a heavy, negative real number and b be a non positive real number. Let us observe that $a + b$ is heavy.

Let a be a non light, positive real number and b be a positive real number. One can check that $a + b$ is heavy.

Let a be a non light, negative real number and b be a negative real number. Let us note that $a + b$ is heavy.

Let a be a non heavy real number and b be a heavy, positive real number. Let us observe that $a + b$ is positive.

Let a be a light real number and b be a non light, positive real number. Note that $a + b$ is positive.

Let a be a non heavy real number. Note that $a + b$ is non negative.

Let b be a heavy, negative real number. Observe that $a + b$ is negative.

Let a be a light real number and b be a non light, negative real number. One can check that $a + b$ is negative.

Let a be a non heavy real number. One can check that $a + b$ is non positive.

Let a be a light, positive real number and c be a light, negative real number. One can verify that $a + c$ is light.

Let a be a non heavy, positive real number and c be a non heavy, negative real number. Let us note that $a+c$ is non heavy.

Let a, b be real numbers. One can check that $a - \min(a,b)$ is non negative.

Let a, b be weightless real numbers. Observe that $\min(a,b)$ is weightless and $\max(a,b)$ is weightless.

Let a, b be light real numbers. Note that $\min(a,b)$ is light and $\max(a,b)$ is light.

Let a, b be heavy real numbers. One can verify that $\min(a,b)$ is heavy and $\max(a,b)$ is heavy.

Let a, b be positive real numbers. Observe that $\frac{\min(a,b)}{\max(a,b)}$ is non heavy and $\frac{\max(a,b)}{\min(a,b)}$ is non light and $\frac{a+b}{a}$ is heavy and $\frac{a}{a+b}$ is light.

Let us consider real numbers a, b. Now we state the propositions:

(15) If $a \cdot b$ is positive, then $|a-b| < |a+b|$.

(16) If $a \cdot b$ is negative, then $|a-b| > |a+b|$.

(17) Let us consider non zero real numbers a, b. Then $|a^2 - b^2| < |a^2 + b^2|$. The theorem is a consequence of (15).

(18) Let us consider positive real numbers a, b, c. If $a < b$, then $\frac{b+c}{a+c}$ is heavy.

(19) Let us consider positive real numbers a, b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \geqslant 1$.

(20) Let us consider negative real numbers a, b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \geqslant 1$.

(21) Let us consider a negative real number a, and a positive real number b. Then $\frac{\frac{a}{b}+\frac{b}{a}}{2} \leqslant -1$.

Let a, b be non zero real numbers. Let us note that $\frac{\frac{a}{b}+\frac{b}{a}}{2}$ is non light and $\frac{a}{b}+\frac{b}{a}$ is heavy.

Now we state the proposition:

(22) Let us consider non zero real numbers a, b. Then $(\frac{a}{b}+\frac{b}{a})^2 \geqslant 4$. The theorem is a consequence of (1).

Let a, b be positive real numbers. Note that $\frac{(a+2\cdot b)\cdot a}{(a+b)^2}$ is non heavy and $\frac{b}{a}+\frac{a}{b}-1$ is non light and $\frac{(a+b)\cdot(a^{-1}+b^{-1})}{4}$ is non light.

Let a, b be light real numbers. Let us note that $\frac{a+b}{1+a\cdot b}$ is non heavy.

Let a, b, c, d be positive real numbers. Note that $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$ is heavy.

Let a be a non negative real number. Observe that $|-a|$ reduces to a.

Observe that there exists a natural number which is trivial and non zero and there exists a natural number which is trivial.

Let a, b be non zero real numbers. One can verify that $\min(a,b)$ is non zero and $\max(a,b)$ is non zero.

Let a be a non negative real number and b be a real number. Let us note that $\max(a, b)$ is non negative.

Let a be a non positive real number. One can check that $\min(a, b)$ is non positive.

Let a be a positive real number. One can verify that $\max(a, b)$ is positive.

Let a be a negative real number. One can verify that $\min(a, b)$ is negative.

Let a, b be non negative real numbers. Observe that $\min(a, b)$ is non negative.

Let a, b be non positive real numbers. One can verify that $\max(a, b)$ is non positive.

Let a be a positive real number and b be a non negative real number. Observe that $\frac{a}{a+b}$ is non heavy and $\frac{a+b}{a}$ is non light.

Let a, b be positive real numbers. One can verify that $\frac{a}{\max(a,b)}$ is non heavy and $\frac{a}{\min(a,b)}$ is non light. Now we state the propositions:

(23) Let us consider real numbers a, b. If $\mathrm{sgn}(a) > \mathrm{sgn}(b)$, then $a > b$.

(24) Let us consider non zero real numbers a, b. Suppose $\mathrm{sgn}(a) > \mathrm{sgn}(b)$. Then

  (i) a is positive, and

  (ii) b is negative.

Let a, b be real numbers. Let us note that $\max(a, b) - \min(a, b)$ is non negative.

One can check that $(\mathrm{sgn}(a - b)) \cdot (\max(a, b) - \min(a, b))$ reduces to $a - b$.

Let a be a real number. Note that a^1 reduces to a and 1^a reduces to 1. One can check that a^0 is natural and a^0 is weightless.

Let a be a positive real number and b be a real number. One can check that a^b is positive.

Let a be a weightless, positive real number and b be a positive real number. Let us note that b^a reduces to b.

Let a be a heavy, positive real number. Observe that a^b is heavy.

Let b be a negative real number. Note that a^b is light.

Let a be a light, positive real number and b be a positive real number. Note that a^b is light.

Let b be a negative real number. Note that a^b is heavy.

Let a be a non weightless, positive real number and b be a real number. Observe that $\log_a(a^b)$ reduces to b.

Let b be a positive real number. Observe that $a^{\log_a b}$ reduces to b.

Now we state the propositions:

(25) Let us consider positive real numbers a, b. Then $a > b$ if and only if $\frac{1}{a} < \frac{1}{b}$.

(26) Let us consider negative real numbers a, b. Then $a > b$ if and only if $\frac{1}{a} < \frac{1}{b}$.

(27) Let us consider positive real numbers a, b. Then $\frac{1}{a} > \frac{1}{b}$ if and only if $-a > -b$.

(28) Let us consider negative real numbers a, b. Then $\frac{1}{a} > \frac{1}{b}$ if and only if $-a > -b$.

(29) Let us consider positive real numbers a, b. Then $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b - a)$.

(30) Let us consider negative real numbers a, b. Then $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b - a)$.

Let us consider non zero real numbers a, b. Now we state the propositions:

(31) $\mathrm{sgn}(\frac{1}{a} - \frac{1}{b}) = \mathrm{sgn}(b - a)$ if and only if $\mathrm{sgn}(b) = \mathrm{sgn}(a)$. The theorem is a consequence of (29), (30), and (24).

(32) $a + b = a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} = 1$.

Let us consider positive real numbers a, b. Now we state the propositions:

(33) $a + b > a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} > 1$.

(34) $a + b < a \cdot b$ if and only if $\frac{1}{a} + \frac{1}{b} < 1$. The theorem is a consequence of (32) and (33).

(35) Let us consider a non heavy, positive real number a, and a positive real number b. Then $a + b > a \cdot b$. The theorem is a consequence of (33).

(36) Let us consider non zero real numbers a, b. Then $a - b = a \cdot b$ if and only if $\frac{1}{b} - \frac{1}{a} = 1$.

(37) Let us consider positive real numbers a, b. If $a - b = a \cdot b$, then b is light. The theorem is a consequence of (1) and (36).

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(38) If $a + b = c + d$, then $\max(a, b) - \max(c, d) = \min(c, d) - \min(a, b)$.

(39) If $a + b = c + d$, then $\max(a, b) = \max(c, d)$ iff $\min(a, b) = \min(c, d)$.

(40) If $a + b = c + d$, then $\max(a, b) > \max(c, d)$ iff $\min(a, b) < \min(c, d)$. The theorem is a consequence of (38).

(41) If $a + b = c + d$ and $a \cdot b = c \cdot d$, then $\max(a, b) = \max(c, d)$. The theorem is a consequence of (38).

Let us consider positive real numbers a, b, c, d and a real number n. Now we state the propositions:

(42) If $a + b = c + d$ and $a \cdot b = c \cdot d$, then $a^n + b^n = c^n + d^n$. The theorem is a consequence of (41).

(43) If $a + b = c + d$ and $a^n + b^n \neq c^n + d^n$, then $a \cdot b \neq c \cdot d$.

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(44) If $a+b=c+d$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}$ iff $a \cdot b=c \cdot d$.

(45) If $a+b=c+d$, then $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$ iff $a \cdot b<c \cdot d$.

(46) If $a+b \geqslant c+d$ and $a \cdot b<c \cdot d$, then $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$.

(47) If $a \cdot b<c \cdot d$ and $\frac{1}{a}+\frac{1}{b} \leqslant \frac{1}{c}+\frac{1}{d}$, then $a+b<c+d$.

(48) If $a+b \leqslant c+d$ and $\frac{1}{a}+\frac{1}{b}>\frac{1}{c}+\frac{1}{d}$, then $a \cdot b<c \cdot d$.

(49) If $a \cdot b \geqslant c \cdot d$, then $a+b>c+d$ or $\frac{1}{a}+\frac{1}{b} \leqslant \frac{1}{c}+\frac{1}{d}$.

(50) Let us consider positive real numbers a, b, and real numbers n, m. Then

(i) $a^{m+n}+b^{m+n}=\frac{(a^m+b^m)\cdot(a^n+b^n)+(a^n-b^n)\cdot(a^m-b^m)}{2}$, and

(ii) $a^{m+n}-b^{m+n}=\frac{(a^m+b^m)\cdot(a^n-b^n)+(a^n+b^n)\cdot(a^m-b^m)}{2}$.

(51) Let us consider positive real numbers a, b, and a real number n. Then $a^{n+1}+b^{n+1}=\frac{(a^n+b^n)\cdot(a+b)+(a-b)\cdot(a^n-b^n)}{2}$. The theorem is a consequence of (50).

Let us consider positive real numbers a, b and positive real numbers n, m. Now we state the propositions:

(52) $a^{n+m}+b^{n+m} \geqslant \frac{(a^n+b^n)\cdot(a^m+b^m)}{2}$.
Proof: $(a^n-b^n)\cdot(a^m-b^m) \geqslant 0$. □

(53) $a^{n+m}+b^{n+m}=\frac{(a^n+b^n)\cdot(a^m+b^m)}{2}$ if and only if $a=b$.
Proof: If $a=b$, then $a^{n+m}+b^{n+m}=\frac{(a^n+b^n)\cdot(a^m+b^m)+0}{2}$. If $a \neq b$, then $(a^n-b^n)\cdot(a^m-b^m)>0$. □

Let us consider positive real numbers a, b, c, d. Now we state the propositions:

(54) If $a+b \leqslant c+d$ and $\max(a,b)>\max(c,d)$, then $a \cdot b<c \cdot d$.

(55) If $a+b \leqslant c+d$ and $a \cdot b>c \cdot d$, then $\max(a,b)<\max(c,d)$ and $\min(a,b)>\min(c,d)$. The theorem is a consequence of (54).

(56) $\max(a,b)=\max(c,d)$ and $\min(a,b)=\min(c,d)$ if and only if $a \cdot b=c \cdot d$ and $a+b=c+d$. The theorem is a consequence of (41).

(57) Let us consider non negative real numbers a, b, and a positive real number c. Then $a \geqslant b$ if and only if $a^c \geqslant b^c$.

(58) Let us consider non negative real numbers a, b, n. Then

(i) $\max(a^n,b^n)=(\max(a,b))^n$, and

(ii) $\min(a^n,b^n)=(\min(a,b))^n$.

The theorem is a consequence of (57).

(59) Let us consider positive real numbers a, b, c, d. Suppose $a \cdot b>c \cdot d$ and $\frac{a}{b} \geqslant \frac{c}{d}$ or $a \cdot b \geqslant c \cdot d$ and $\frac{a}{b}>\frac{c}{d}$. Then $a>c$.

(60) Let us consider a positive real number a. Then $1-a<\frac{1}{1+a}$.

(61) Let us consider a light, positive real number a. Then $1 + a < \frac{1}{1-a}$.

(62) Let us consider positive real numbers a, b, a non negative real number m, and a positive real number n. If $a^m + b^m \leqslant 1$, then $a^{m+n} + b^{m+n} < 1$. The theorem is a consequence of (1).

(63) Let us consider positive real numbers a, b, a non positive real number m, and a negative real number n. If $a^m + b^m \leqslant 1$, then $a^{m+n} + b^{m+n} < 1$. The theorem is a consequence of (62).

(64) Let us consider positive real numbers a, b, c, n, and a non negative real number m. If $a^m + b^m \leqslant c^m$, then $a^{m+n} + b^{m+n} < c^{m+n}$. The theorem is a consequence of (62).

(65) Let us consider positive real numbers a, b, and a heavy, positive real number n. Then $a^n + b^n < (a+b)^n$. The theorem is a consequence of (64).

Let k be a positive real number and n be a heavy, positive real number. Let us observe that $(k+1)^n - k^n$ is heavy and positive.

Let k be a heavy, positive real number and n be a non negative real number. One can verify that $k^{n+1} - k^n$ is positive.

Now we state the propositions:

(66) Let us consider a positive real number k, and a heavy, positive real number n. Then $(k+1)^n > k^n + 1$. The theorem is a consequence of (65).

(67) Let us consider positive real numbers a, b, and a light, positive real number n. Then $a^n + b^n > (a+b)^n$. The theorem is a consequence of (64).

(68) Let us consider a positive real number k, and a light, positive real number n. Then $(k+1)^n < k^n + 1$. The theorem is a consequence of (67).

(69) Let us consider a positive real number k, and a non positive real number n. Then $(k+1)^n < k^n + 1$.

(70) Let us consider positive real numbers a, b, and a non positive real number n. Then $a^n + b^n > (a+b)^n$. The theorem is a consequence of (69).

Let us consider positive real numbers a, b and a real number n. Now we state the propositions:

(71) $(a+b)^n > a^n + b^n$ if and only if n is heavy and positive. The theorem is a consequence of (1), (67), (70), and (65).

(72) $(a+b)^n = a^n + b^n$ if and only if $n = 1$. The theorem is a consequence of (71), (70), and (67).

(73) $(a+b)^n < a^n + b^n$ if and only if $n < 1$. The theorem is a consequence of (1), (71), and (72).

Let us consider positive real numbers a, b, c. Now we state the propositions:

(74) $(a+b) \cdot (a+c) > a \cdot (a+b+c)$.

(75) $\frac{a+b+c}{a+b} < \frac{a+c}{a}$. The theorem is a consequence of (74).

(76) Let us consider positive real numbers a, b, c, and a positive real number n. Then $\frac{(a+b+c)^n}{(a+b)^n} < \frac{(a+c)^n}{a^n}$. The theorem is a consequence of (75).

(77) Let us consider heavy, positive real numbers a, b. Then $a+b-1 > \frac{a}{b} > \frac{1}{a+b-1}$. The theorem is a consequence of (1).

(78) Let us consider positive real numbers a, b, c. Then $\frac{a+b+c}{a} > \frac{a+b}{a+c} > \frac{a}{a+b+c}$. The theorem is a consequence of (77).

Let us consider a light, positive real number a and a heavy, positive real number n. Now we state the propositions:

(79) $(1+a)^n \cdot (1-a)^n < (1+a^n) \cdot (1-a^n)$. The theorem is a consequence of (65).

(80) $\frac{(1+a)^n}{1+a^n} < \frac{1-a^n}{(1-a)^n}$. The theorem is a consequence of (79).

Let us consider a light, positive real number a. Now we state the propositions:

(81) (i) $\max(a, 1-a) \geqslant \frac{1}{2}$, and

(ii) $\min(a, 1-a) \leqslant \frac{1}{2}$.

(82) $\frac{1}{1+a} + \frac{1}{1-a} > 2$.

(83) Let us consider a heavy, positive real number a. Then $\frac{1}{a+1} + \frac{1}{a-1} > \frac{2}{a}$.

(84) Let us consider positive real numbers a, b, and a heavy, positive real number n. Then $(2 \cdot a+b)^n + b^n < 2 \cdot (a+b)^n$. The theorem is a consequence of (65).

(85) Let us consider heavy, positive real numbers a, n. Then $(a+1)^n - (a-1)^n > 2^n$. The theorem is a consequence of (65).

(86) Let us consider a light, positive real number a, and a heavy, positive real number n. Then $2^n > (1+a)^n - (1-a)^n > 2 \cdot a^n$. The theorem is a consequence of (1) and (65).

(87) Let us consider heavy, positive real numbers a, n, and a light, positive real number b. Then $(a+1)^n - (a-1)^n > (a+b)^n - (a-b)^n > 2 \cdot b^n$. The theorem is a consequence of (1) and (65).

(88) Let us consider positive real numbers a, b, and a positive real number n. Then $2 \cdot (a+b)^n > (a+b)^n + a^n > 2 \cdot (a^n)$.

Let us consider positive real numbers a, b. Now we state the propositions:

(89) If $a \neq b$, then there exist real numbers n, m such that $a = \frac{a}{b}^n$ and $b = \frac{a}{b}^m$.

(90) If $a \neq b$, then there exist real numbers n, m such that $a - b = \frac{a}{b}^n \cdot (\frac{a}{b}^m - 1)$. The theorem is a consequence of (89).

(91) Let us consider positive real numbers a, m, n. Then $a^n + a^m = a^{\min(n,m)} \cdot (1 + a^{|m-n|})$.

(92) Let us consider non weightless, positive real numbers a, b. Then $\log_a b = \frac{1}{\log_b a}$. The theorem is a consequence of (1).

Let a be a heavy, positive real number and b be a positive real number. One can check that $\log_a(a+b)$ is heavy and $\log_{a+b} a$ is light.

Now we state the propositions:

(93) Let us consider a positive, non weightless real number a, and a positive real number b. Then $\log_a b = 0$ if and only if $b = 1$.
PROOF: $|a| \neq 1$. If $\log_a b = 0$, then $b = 1$. □

(94) Let us consider a non weightless, positive real number a, and a positive real number b. Then $\log_a b = 1$ if and only if $a = b$. The theorem is a consequence of (1).

(95) Let us consider positive real numbers a, b, and a non zero real number n. Then $a^n = b^n$ if and only if $a = b$.
PROOF: If $a \neq b$, then $a^n \neq b^n$. □

(96) Let us consider a non weightless, positive real number a, and a positive real number b. Then

(i) $\log_a b = -\log_{\frac{1}{a}} b$, and

(ii) $\log_{\frac{1}{a}} b = \log_a \frac{1}{b}$, and

(iii) $\log_a b = -\log_a \frac{1}{b}$, and

(iv) $\log_a b = \log_{\frac{1}{a}} \frac{1}{b}$.

The theorem is a consequence of (1).

(97) Let us consider a heavy, positive real number a, and a positive real number b. Then $a > b$ if and only if $\log_a b < 1$.
PROOF: $a > 1$. If $\log_a b < 1$, then $a > b$. If $a > b$, then $\log_a b < 1$. □

(98) Let us consider a light, positive real number a, and a positive real number b. Then $a < b$ if and only if $\log_a b < 1$. The theorem is a consequence of (97) and (96).

(99) Let us consider a heavy, positive real number a, and a positive real number b. Then $a < b$ if and only if $\log_a b > 1$. The theorem is a consequence of (97) and (94).

(100) Let us consider a light, positive real number a, and a positive real number b. Then $a > b$ if and only if $\log_a b > 1$. The theorem is a consequence of (99) and (96).

Let us consider non weightless, positive real numbers a, b. Now we state the propositions:

(101) If $\log_a b \geqslant 1$, then $0 < \log_b a \leqslant 1$. The theorem is a consequence of (92).

(102) If $\log_a b \leqslant -1$, then $0 > \log_b a \geqslant -1$. The theorem is a consequence of (92).

Let us consider heavy, positive real numbers a, b. Now we state the propositions:

(103) If $\log_a b > \log_b a \geqslant 1$, then $a > b$. The theorem is a consequence of (1).

(104) If $\log_b a < 1$, then $a < b$. The theorem is a consequence of (1) and (94).

Let us consider heavy, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(105) If $\log_a b \leqslant \log_c d$ and $a < b$, then $c < d$. The theorem is a consequence of (99).

(106) If $\log_a b \geqslant \log_c d$ and $a > b$, then $c > d$. The theorem is a consequence of (97).

Let us consider a heavy, positive real number a, a light, positive real number c, and positive real numbers b, d. Now we state the propositions:

(107) If $\log_a b \leqslant \log_c d$ and $a < b$, then $c > d$. The theorem is a consequence of (99) and (100).

(108) If $\log_a b \geqslant \log_c d$ and $a > b$, then $c < d$. The theorem is a consequence of (97) and (98).

Let us consider light, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(109) If $\log_a b \leqslant \log_c d$ and $a > b$, then $c > d$. The theorem is a consequence of (96) and (105).

(110) If $\log_a b \geqslant \log_c d$ and $a < b$, then $c < d$. The theorem is a consequence of (96) and (106).

Let us consider a light, positive real number a, a heavy, positive real number c, and positive real numbers b, d. Now we state the propositions:

(111) If $\log_a b \leqslant \log_c d$ and $a > b$, then $c < d$. The theorem is a consequence of (100) and (99).

(112) If $\log_a b \geqslant \log_c d$ and $a < b$, then $c > d$. The theorem is a consequence of (98) and (97).

Let us consider heavy, positive real numbers a, c and positive real numbers b, d. Now we state the propositions:

(113) If $\log_a b < \log_c d$ and $a \leqslant b$, then $c < d$. The theorem is a consequence of (97) and (99).

(114) If $\log_a b \leqslant \log_c d$ and $a \leqslant b$, then $c \leqslant d$. The theorem is a consequence of (97).

(115) Let us consider positive real numbers a, b. If $a > b$, then $\log_{\frac{a}{b}} a > \log_{\frac{a}{b}} b$.

Acknowledgement: Ad Maiorem Dei Gloriam

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Accepted May 31, 2020